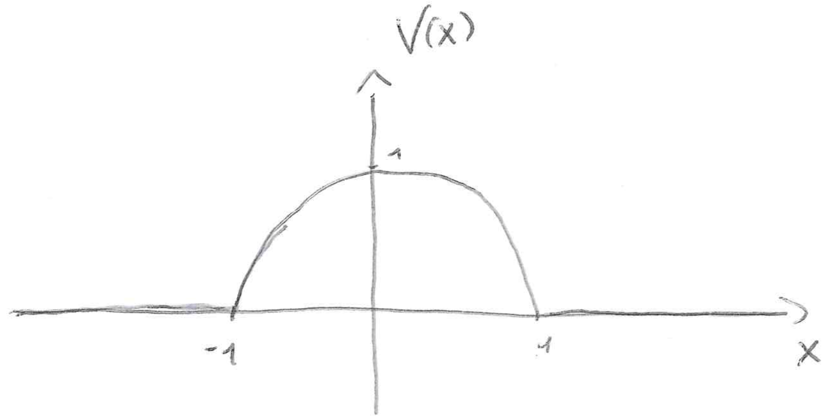
a)



b)

$E = \frac{p^2}{2m} + V(x) \geq V(x)$ ↦ classically allowed regions.

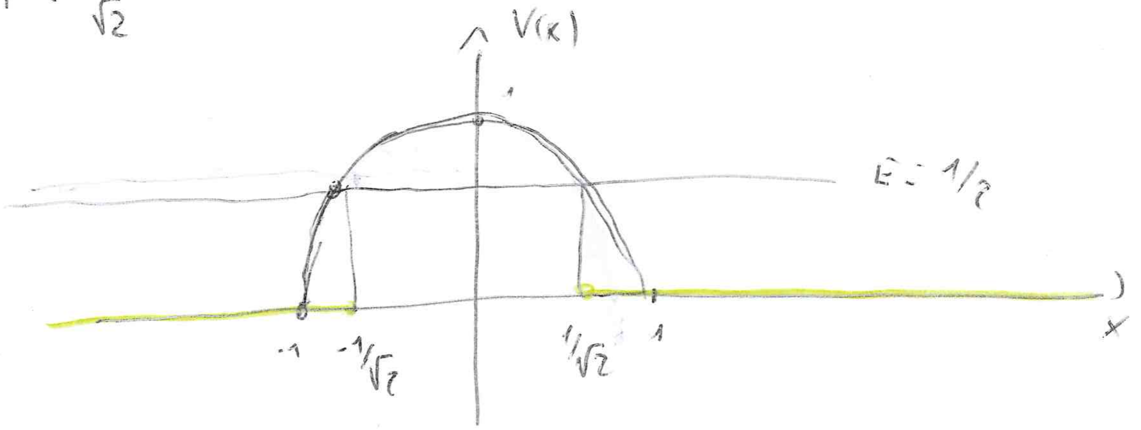
Therefore:

$$E \geq V(x) \rightarrow \frac{1}{2} \geq \begin{cases} -x^2+1 & |x|<1 \\ 0 & |x|>1 \end{cases} \rightarrow \frac{1}{2} \geq -x^2+1$$

$x^2 \geq \frac{1}{2}$

$|x| \geq \frac{1}{\sqrt{2}}$ classically allowed

$|x| < \frac{1}{\sqrt{2}}$ " " forbidden



c)

$$T = e^{-\frac{2}{\hbar}\int_{x_A}^{x_B}\sqrt{2m\,(V(x)-E)}}$$ whereas $V(x_A) = V(x_B) = E$.

In our case:

$$T = e^{-\frac{2}{\hbar}\int_{-1/\sqrt{2}}^{1/\sqrt{2}}\sqrt{2m\,(-x^2+1-\frac{1}{2})}} = e^{-\frac{2}{\hbar}\int_{-1/\sqrt{2}}^{1/\sqrt{2}}\sqrt{2m\left(\frac{1}{2}-x^2\right)}}$$

Being:

$$\int\sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\arctan\left(\frac{x}{\sqrt{a^2-x^2}}\right)$$

We get (for $a^2 = 1/2$):

$$T = e^{-\frac{2}{\hbar}\sqrt{2m}\left(\frac{1}{4}\operatorname{arctg}(\infty) - \frac{1}{4}\operatorname{arctg}(-\infty)\right)} = e^{-\frac{2}{\hbar}\sqrt{2m}\frac{\pi}{4}} = e^{-\frac{\sqrt{m}}{\hbar}\frac{\pi}{\sqrt{2}}}.$$

$$\boxed{T = e^{-\frac{\sqrt{m}}{\hbar}\frac{\pi}{\sqrt{2}}}}$$

In fact:

$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}}\sqrt{\frac{1}{2}-x^2}\,dx = \left[\frac{x\sqrt{\frac{1}{2}-x^2}}{2} + \frac{1}{4}\operatorname{arctg}\left(\frac{x}{\sqrt{\frac{1}{2}-x^2}}\right)\right]_{-1/\sqrt{2}}^{1/\sqrt{2}} =$$

$$= \left[0 + \frac{1}{4}\operatorname{arctg}\left(\frac{1/\sqrt{2}}{0^+}\right) - 0 - \frac{1}{4}\operatorname{arctg}\left(\frac{-1/\sqrt{2}}{0^+}\right)\right]$$

$$= \frac{1}{2}\operatorname{arctg}(\infty) = \frac{\pi}{4}$$

$$\Rightarrow \quad Z = \frac{1}{N!}\left(\frac{V}{\Lambda^3}\right)^N = \frac{1}{N!}\left(\frac{V}{h^3}(2\pi m K_B)^{3/2} T^{3/2}\right)^N$$

- $$\boxed{\mathcal{E} = K_B T^2 \frac{\partial \ln Z}{\partial T}}$$

$$\ln Z = N \ln\left(V\left(\frac{\sqrt{2\pi m K_B}}{h}\right)^3 T^{3/2}\right) - \ln N!$$

$$\frac{\partial \ln Z}{\partial T} = N \frac{3}{2} T^{1/2} \cdot \frac{1}{T^{3/2}} = \frac{3}{2}\frac{N}{T}$$

Ergo:

$$\boxed{\mathcal{E} = \frac{3}{2} N K_B T.}$$

- $$\boxed{P = K_B T \frac{\partial \ln Z}{\partial V} = \frac{N K_B T}{V}}$$

- $$S = K_B \ln Z + K_B T \frac{\partial \ln Z}{\partial T} = K_B N \ln\left(V\left(\frac{\sqrt{2\pi m K_B}}{h}\right)^3 T^{3/2}\right) - K_B \ln N! + \frac{3}{2} N K_B.$$

b) $Z = Z_A \cdot Z_B$

whereas

$$Z_A = \frac{1}{N_A!}\left(\frac{V_A}{h^3}(2\pi m K_B)^{3/2} T_A^{3/2}\right)^{N_A}$$

$$Z_B = \frac{1}{N_B!}\left(\frac{V_B}{h^3}(2\pi m K_B)^{3/2} T_B^{3/2}\right)^{N_B}$$

$\mathcal{E} = \mathcal{E}_A + \mathcal{E}_B$ with $\mathcal{E}_A = \frac{3}{2} N_A T_A K_B$, $\mathcal{E}_B = \frac{3}{2} N_B T_B K_B$.

c) $\mathcal{E}_A = \frac{3}{2} N_A T_A K_B$, $\mathcal{E}_B = \frac{3}{2} N_B T_B K_B$

$\mathcal{E}_{initial} = \frac{3}{2}(N_A T_A + N_B T_B) K_B$ is the 'initial' energy.

When equilibrium is reached we get:

$\mathcal{E}_{final} = \frac{3}{2}(N_A + N_B) K_B T_{eq}$

By writing $\mathcal{E}_{initial} = \mathcal{E}_{final}$ we get:

$$\boxed{T = \frac{N_A T_A + N_B T_B}{N_A + N_B}}$$

a) $\begin{cases} |E_1\rangle & \text{with eigenvalue } E_1. \\ |E_2\rangle & \text{" " } E_2. \end{cases}$

b) siehe später

c)

$$|s(t)\rangle = e^{-iHt/\hbar}\left(\frac{1}{\sqrt{2}}|E_1\rangle - \frac{1}{\sqrt{2}}|E_2\rangle\right) =$$

$$= \frac{e^{-iE_1t/\hbar}}{\sqrt{2}}|E_1\rangle - \frac{e^{-iHt/\hbar}}{\sqrt{2}}|E_2\rangle.$$

d)

$$a(t) = \langle U|s(t)\rangle =$$

$$= \left(\frac{1}{\sqrt{2}}\langle E_1| + \frac{1}{\sqrt{2}}\langle E_2|\right)\left(\frac{e^{-iE_1t/\hbar}}{\sqrt{2}}|E_1\rangle - \frac{e^{-iHt/\hbar}}{\sqrt{2}}|E_2\rangle\right)$$

$$= \frac{1}{2}e^{-iE_1t/\hbar} - \frac{1}{2}e^{-iE_2t/\hbar}$$

$$p(t) = |a(t)|^2 = \frac{1}{4} + \frac{1}{4} - \frac{1}{4}e^{+i\Delta E t/\hbar} - \frac{1}{4}e^{-i\Delta E t/\hbar}, \quad (\text{where } \Delta E = E_2 - E_1$$

$$\boxed{p(t) = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{\Delta E}{\hbar}\cdot t\right)}$$

a)

$$\begin{cases} P(t) = 0 & \text{for} \quad t = 2m\pi \cdot \dfrac{\hbar}{\Delta E} \\[2em] P(t) = 1 & \text{for} \quad t = (2m+1)\pi \, \dfrac{\hbar}{2E} \end{cases}$$

b) $A = \alpha |E_1\rangle\langle E_2|$ is not Hermitian.

In fact:

$$A^\dagger = \alpha^* |E_2\rangle\langle E_1| \neq A.$$

$$A + A^\dagger = \alpha |E_1\rangle\langle E_2| + \alpha^* |E_2\rangle\langle E_1|$$

(which is by the way hermitian).

a)

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right) - \frac{\hbar\omega}{2} = \hbar\omega n \qquad n = 0, 1, \dots$$

b)

$$Z = \sum_{n=0}^{\infty} e^{-\beta E_n} = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} = \sum_{n=0}^{\infty} \left(e^{-\beta\hbar\omega}\right)^n =$$

$$= \frac{1}{1-e^{-\beta\hbar\omega}} .$$

qed.

c) $\hat{\varrho}$ is given by the following matrix elements:

$$\hat{\varrho}_{mn} = \langle m|\hat{\varrho}|n\rangle = \frac{e^{-\beta E_n}}{Z}\delta_{mn} = \frac{e^{-\beta\hbar\omega\cdot n}}{\frac{1}{1-e^{-\beta\hbar\omega}}}\delta_{mn}$$

$$= \left(1-e^{-\beta\hbar\omega}\right)\cdot e^{-\beta\hbar\omega\cdot n}\,\delta_{mn}$$

other possibility to write it:

$$\hat{\varrho} = \sum_{n=0}^{\infty} P_n |n\rangle\langle n| \qquad \text{with} \quad P_n = \left(1-e^{-\beta\hbar\omega}\right) e^{-\beta\hbar\omega\cdot n}$$

d) $$Z_N(T,N) = \frac{Z(T)^N}{N!} \quad \text{with} \quad Z(T) = \frac{1}{1-e^{-\beta\hbar\omega}}$$

e) $$\mathcal{E} = -\partial_\beta \ln Z(T,N) = -N\partial_\beta \ln\left(\frac{1}{1-e^{-\beta\hbar\omega}}\right) =$$

$$= N\partial_\beta \ln\left(1-e^{-\beta\hbar\omega}\right) =$$

$$= N(-1)(-\hbar\omega)e^{-\beta\hbar\omega}\frac{1}{1-e^{-\beta\hbar\omega}}$$

$$\boxed{\mathcal{E} = N\frac{\hbar\omega}{e^{\beta\hbar\omega}-1}}$$

f) $$Z(T,\mu) = \sum_{N=0}^{\infty} e^{\beta\mu N}\frac{Z(T)^N}{N!} = \sum_{N=0}^{\infty}\frac{\left(e^{\beta\mu}Z(T)\right)^N}{N!} =$$

$$= e^{e^{\beta\mu}Z(T)}$$