Bell & GHZM

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Abstract

Simple examples about the differences between classical and quantum theories. The Bell inequalities are discussed in a particular but significant context involving photons. The GHZM experiment is presented.

1 Gedankenexperiment with photons

1.1 Quantum and classical correlations

Two photons are emitted back to back along the x-axis. The first photon goes to the left ($x < 0$), the second to the right ($x > 0$). Let us consider the following polarization:

$$|S\rangle = \sqrt{\frac{1}{2}}\left(|HH\rangle + |VV\rangle\right) \tag{1}$$

where H stands for horizontal (i.e. the y-axis) and V for vertical (i.e. the z-axis).

What happen if on both sides we put a polarizer in the H direction?

The answer is easy: in the 50% of cases both photons go through ($|HH\rangle$) and in the 50% of cases they are absorbed ($|VV\rangle$).

According to the Copenhagen interpretation, after that -say- both photons go through, the wave function collapses instantaneously to $|HH\rangle$:

$$|S\rangle \to |HH\rangle\,. \tag{2}$$

Note that we can imagine that both photons are very far apart when they are "measured" by the polarizers. If, however, HH or VV is realized, is decided only in the act of the measurement and not before. Interestingly, this collapse happens instantaneously even if the two photons are now space-like separated. This is the "spooky action at distance" that Einstein referred to. It is however important to stress that this instantaneous collapses does NOT allow to send information. The speed of light remains the upper limit for that. In this sense we can say that QM and special relativity coexist peacefully.

One can think that all this is crazy. In fact, one may argue as follows: the photons, in the moment of their creation, already come out as $|HH\rangle$ in the 50% of cases, and $|VV\rangle$ in the other 50% [1] Then,

[1] One could imagine the following situation: in the source there is a chaotic pendulum oscillating. At each second the two photons are emitted as HH if the pendulum-ball is on the left side, VV otherwise. This system is perfectly deterministic, although it is practically impossible to predict the polarizations because of the chaotic nature of the system. This hypothetical chaotic pendulum plays the role of the 'hidden variable' in this context. As we shall see, Nature is not like that: there is no secrete 'chaotic pendulum' or something of that kind in the photon source.

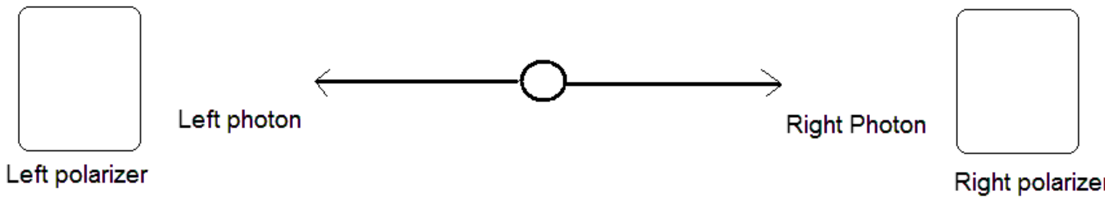

Figure 1: Schematic setup of the experiment. One can chose how to rotate each polarizer.

when we measure H on the left, we know also that the other photon is H, but no "collapse" of the wave function took place. This kind of correlation is a classical correlation of the type of Bertlmann socks: Mr. Bertlmann puts on every morning a couple of red socks or a couple of green socks. The choice happens randomly, but when we see "i.e. measure" that on the left foot he carries a red sock, we also know that on the right foot there is also a red sock.

Can we distinguish between the quantum and the classical correlations? At this level we cannot. These two pictures explain equally well the 50% HH and the 50% VV, which we measure with the polarizers. In order to find a difference we have to perform different kind of measurements on the left and on the right photons. We thus tilt the left polarizer of an angle θ_1 with respect to the horizontal (y) direction, and similarly we tilt the right polarizers of an angle θ_2. In order to properly understand what is going on we first discuss the change of basis connected with the rotation of the polarizer.

For references see [1, 2, 3] and refs. therein.

1.2 Tilting the polarizers

Let us forget for a moment about the two entangled photons We consider a single photon which is H-polarized and travels along the x-axis to the left polarizer. Now, we rotate this polarizer by an angle θ_1 (D_1 direction). Does it go through or not? Well, we can calculate probabilities. To this end we have to perform a change of basis. We can express $|H\rangle$ and $|V\rangle$ in term of the polarization along the D_1 and the D_1' directions, where D_1' is orthogonal to D_1.

Mathematically:

$$\begin{pmatrix} |H\rangle \\ |V\rangle \end{pmatrix} = \begin{pmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{pmatrix} \begin{pmatrix} |D_1\rangle \\ |D_1'\rangle \end{pmatrix} . \tag{3}$$

where $c_1 = \cos\theta_1$ and $s_1 = \sin\theta_1$.

That is:

$$|H\rangle = c_1 |D_1\rangle - s_1 |D_1'\rangle . \tag{4}$$

Now, the probability that the H-polarized photon goes through the polarizers set in the D_1 direction is $p = \cos^2\theta_1$ (Malus law). Vice-versa, if we have a vertically polarized photons travelling into the polarizer, we would have a probability $\sin^2\theta_1$ that it goes through.

Similarly, we now consider a single photon travelling along the x-axis to the right. it is measured by a polarizer set in the D_2 direction, tilted by an angle θ_2 with respect to the horizontal (y) axis. Everything as before. We express also $|H\rangle$ and $|V\rangle$ along the D_2 and D_2' basis, where D_2' is orthogonal to D_2 :

$$\begin{pmatrix} |H\rangle \\ |V\rangle \end{pmatrix} = \begin{pmatrix} c_2 & -s_2 \\ s_2 & c_2 \end{pmatrix} \begin{pmatrix} |D_2\rangle \\ |D_2'\rangle \end{pmatrix}. \tag{5}$$

where $c_2 = \cos\theta_2$ and $s_2 = \sin\theta_2$. That is, if the photon is horizontally polarized, we have a $c_2^2 = \cos^2\theta_2$ probability that it goes through.

1.3 Classical result without entanglement

We come back to our experiment. Now, the left polarizers is along the D_1 direction and the right polarizer is along the D_2 direction.

Let us follow the classical thinking: This stuff of entanglement and wave-function collapse through the spooky action is crazy. There is no state like $|S\rangle = \sqrt{\frac{1}{2}}\left(|HH\rangle + |VV\rangle\right)$. The photons already come out 50% of times as HH and 50% of times as VV. Then, which is the probability that they both are transmitted? Easy:

$$P_{T,kl} = \frac{1}{2}c_1^2c_2^2 + \frac{1}{2}s_1^2s_2^2. \tag{6}$$

Similarly, the probability that they are both absorbed is

$$P_{A,kl} = \frac{1}{2}s_1^2s_2^2 + \frac{1}{2}c_1^2c_2^2. \tag{7}$$

Summarizing, the probability that both photons show the same behavior (i.e. both through or both stopped) is:

$$K_{kl} = P_{T,kl} + P_{A,kl} = c_1^2c_2^2 + s_1^2s_2^2 \tag{8}$$

In the interesting case in which $\theta_2 = -\theta_1 = -\theta$ we get

$$K_{kl} = \cos^4\theta + \sin^4\theta = 1 - 2\sin^2\theta\cos^2\theta. \tag{9}$$

Well, this is a clear and testable result. What does QM say?

1.4 QM result with entanglement

We take QM seriously. Our state is $|S\rangle = \sqrt{\frac{1}{2}}\left(|HH\rangle + |VV\rangle\right)$. In order to calculate the probabilities of transmission and absorption we first have to rewrite it in the appropriate basis. The first (left) photon is rewritten in terms of the D_1, D_1' basis, while the second (right) photon is rewritten in the D_2, D_2' basis. We obtain the following state:

$$|S\rangle = \sqrt{\frac{1}{2}}\left(|HH\rangle + |VV\rangle\right) = \tag{10}$$

$$= \sqrt{\frac{1}{2}}\left[\left(c_1|D_1\rangle - s_1|D_1'\rangle\right)\left(c_2|D_2\rangle - s_2|D_2'\rangle\right) + \left(s_1|D_1\rangle + c_1|D_1'\rangle\right)\left(s_2|D_2\rangle + c_2|D_2'\rangle\right)\right] \tag{11}$$

By a simple evaluation:

$$\begin{aligned} |S\rangle = {} & \sqrt{\frac{1}{2}}\left(c_1c_2|D_1D_2\rangle + s_1s_2|D_1'D_2'\rangle - c_1s_2|D_1D_2'\rangle - s_1c_2|D_1'D_2\rangle\right) \\ & + \sqrt{\frac{1}{2}}\left(s_1s_2|D_1D_2\rangle + c_1c_2|D_1'D_2'\rangle + s_1c_2|D_1D_2'\rangle + c_1s_2|D_1'D_2\rangle\right). \end{aligned} \tag{12}$$

Now, the (quantum) probability that they are both transmitted is given by

$$P_{T,qm} = \frac{1}{2}\left(c_1c_2 + s_1s_2\right)^2 \tag{13}$$

while the prob. that they are both absorbed is

$$P_{A,qm} = \frac{1}{2}\left(c_1c_2 + s_1s_2\right)^2. \tag{14}$$

Thus, the coincidence (both absorbed or both through) is given by

$$K_{qm} = \left(c_1c_2 + s_1s_2\right)^2. \tag{15}$$

In the particular case of $\theta_2 = -\theta_1 = -\theta$ we get

$$K_{qm} = \cos^2 2\theta. \tag{16}$$

1.5 Comparison

When both polarizers are in the same direction we could not distinguish among the classical and the quantum cases. Now we have obtained two predictable and different result, which we can compare and test. In Fig. 2 the plot is shown.

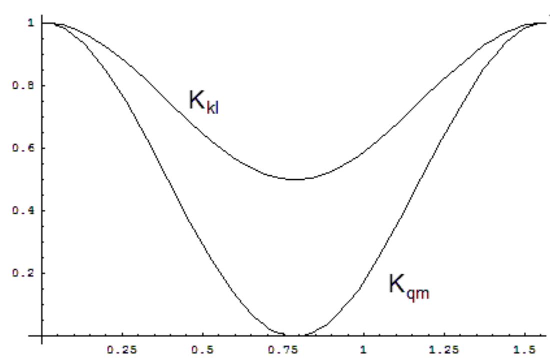

Figure 2: Classical vs quantum coincidences.

Note, they coincide (by construction) for $\theta = 0$. But then, they do not for other values of θ. Needless to say, QM has been verified and the here presented naive classical model rejected.

1.6 Is a classical model possible?

We may think: ok, the here presented classical model was too naive. Maybe we could conceive a better model which gives the same prediction of the quantum result for each value of θ_1 and θ_2 of the inclinations of the two polarizers. The important point is that this is not possible. Whatever is our classical model of production of photons (such as the already cited chaotic pendulum), it is not possible that it reproduces the result of quantum mechanics. In other words, no classical model (based on local realism) can predict the quantum result of Eq. (15).

In fact, each theory based on local realism needs to fulfill the following bound:

$$K_{kl} \geq \left|1 - 2\sin^2\theta\right| \tag{17}$$

This is an example of a Bell inequality.

In order to prove it, we suppose that there is indeed a classical theory which reproduces all the quantum results. That is, the coincidence rate must coincide with the quantum result:

$$K_{qm}(\theta_1, \theta_2) = (c_1 c_2 + s_1 s_2)^2 . \tag{18}$$

In a local theory we have the following situation. The photons are emitted by the source and they "already" know "what to do" when a polarization is measured. That is, for each left photon there is an unknown function of the type $L(\theta)$, where $L(\theta) = 1$ for some values of θ and -1 for the other values. If we perform a measure of the polarization of the left photon along the direction θ, we are actually measuring the function $L(\theta)$. If $L(\theta) = 1$ the photon goes through, if $L(\theta) = -1$ the photon is absorbed. For the right photon there is an analogous function $R(\theta)$.

Now, when the photon pair is emitted, the left photon carries the function $L(\theta)$ while the right photon carries the function $R(\theta)$. However, we should not forget that we are repeating the experiment many times. That is, when the k-pair is emitted, the left photon carries the function $L_k(\theta)$ while the right photon carries the function $R_k(\theta)$. Now, if all these functions were known (and we suppose that there is some unknown way to calculate them) it would be perfectly possible to predict what it will happen at each measurement. (Note, the functions $R_k(\theta)$ and $L_k(\theta)$, but whatever complicated they might be, they are in principle deterministic. One could imagine to have a chaotic pendulum for each θ inside a the source, i.e. an infinity of that. Already this picture is rather crazy, but the very point is that is can be falsified).

Now, let us suppose that we measure $\theta_1 = \theta_2 = 0$. In this case the quantum result is $K = 1$. That means, if for our hypothetical left photon we have:

$$LP(\theta = 0) = \{L_1(\theta = 0), L_2(\theta = 0), ...\} = \{1, -1, 1, 1, -1, 1, -1, ...\} \tag{19}$$

then the right photon must be equal

$$RP(\theta = 0) = \{R_1(\theta = 0), R_2(\theta = 0), ...\} = \{1, -1, 1, 1, -1, 1, -1, ...\}. \tag{20}$$

In this way the perfect coincidence is also classically explained.

Then, let us consider $\theta_1 = \theta$, $\theta_2 = 0$. The quantum coincidence is $K = \cos^2\theta$. That is, a $\sin^2\theta$ fraction of $L_k(\theta)$ changed sign and thus do not coincide with the original string $R_k(0) = L_k(0)$ any longer. (One defines the mismatch as $M = 1 - K = \sin^2\theta$.)

If we consider $\theta_1 = 0$, $\theta_2 = -\theta$ the quantum coincidence is still $K = \cos^2\theta$. Also here a $\sin^2\theta$ fraction of $R_k(-\theta)$ changed sign and thus do not coincide with $L_k(0) = R_k(0)$ any longer. (The mismatch is also here equal to $M = 1 - K = \sin^2\theta$.)

Now, what happens if we consider $\theta_1 = \theta$, $\theta_2 = -\theta$? We realize that the mismatch cannot exceed $\sin^2\theta + \sin^2\theta = 2\sin^2\theta$. In fact

$$M = 2\sin^2\theta - k \tag{21}$$

where k is the fraction of numbers of the lists $\{L_k(\theta)\}$ and $\{R_k(-\theta)\}$ which have changed sign in the same place with respect to $L_k(0) = R_k(0)$ and thus coincide again. That is we can conclude that, whatever is our local classical theory, we get:

$$M \leq 2\sin^2\theta. \tag{22}$$

(This equality takes place when $k = 0$, that is when the change of sign in one list is never accompanied by the change of sign in the other list). That is

$$K_{kl} \geq 1 - 2\sin^2\theta. \tag{23}$$

The result is actually meaningful only for $\theta \leq \pi/4$. It could have also been been obtained by noticing that at least $2(\cos^2\theta - \frac{1}{2})$ must coincide. $K_{kl} \geq 2\cos^2\theta - 1$.

When $\theta \geq \pi/4$ a similar reasoning leads us to conclude that $K_{kl} \geq 2\cos^2\theta - 1$. The results can be summarized in the Bell inequalities

$$K_{kl} \geq \left|1 - 2\sin^2\theta\right| = K_{Bell} \tag{24}$$

In Fig. 3 the Bell-inequalities is also depicted. While it is evident that the previously calculated classical result is above the Bell bound (as it must) the QM result is actually always below! No classical local theory can reproduce this quantum result, no matter how crazy and complicated the functions $L_k(\theta)$ and $R_k(\theta)$ might be.

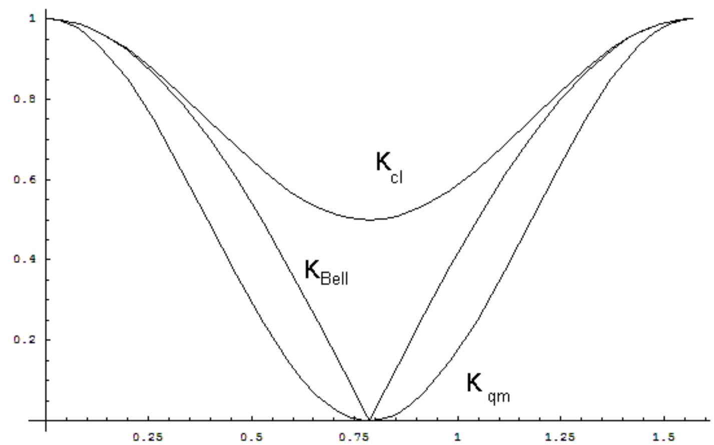

Figure 3: Comparsion of classical and qm coincidences in relation to the Bell-bound.

1.7 Formal Proof of the Bell inequalities

For simplicity we refer to the here analyzed case of photons. Let λ be a ‘hidden’ parameter, whose knowledge would specify if the left photon goes through a polarizer in the θ_1 direction and the right photon through a polarizer in the θ_2 direction. We have thus the functions $L(\theta, \lambda)$ and $R(\theta, \lambda)$, which are perfectly known if λ is given.

As before, the functions L and R are either $+1$ or -1, according to the values of θ (at a specified value of λ).

Now, when the photons are emitted there must be some chaotic system which specifies the value of λ. We do not know exactly how this chaotic system looks like, but there must be a probability distribution $p(\lambda)$. The quantity $p(\lambda)d\lambda$ is thus the probability that -for a given back-to-back photon emission- a value of λ between λ and $\lambda + d\lambda$ is realized. Be for simplicity $0 \leq \lambda \leq 1$.

The coincidence function $K(\theta_1, \theta_2)$ can be expressed in the following way:

$$K(\theta_1, \theta_2) = \int_0^1 d\lambda L(\theta_1, \lambda) R(\theta_2, \lambda) p(\lambda). \tag{25}$$

Now, when $\theta_1 = \theta_2 = \theta$ it must be that

$$R(\theta, \lambda) = L(\theta, \lambda). \tag{26}$$

This is necessary in order that our classical local theory reproduces the quantum result $K_{qm}(\theta_1, \theta_2) = (c_1 c_2 + s_1 s_2)^2$ for $\theta_1 = \theta_2$, for which $K_{qm} = 1$.

In fact, the condition $R(\theta, \lambda) = L(\theta, \lambda)$ implies that, as desired,

$$K(\theta_1 = \theta, \theta_2 = \theta) = \int_0^1 d\lambda L^2(\theta, \lambda) p(\lambda) = \int_0^1 d\lambda p(\lambda) = 1 \tag{27}$$

where the property $L^2(\theta, \lambda) = 1$ has been used.

Let us know consider

$$K(\theta_1, \theta_2) + K(\theta_1, \theta_3) = \int_0^1 d\lambda L(\theta_1, \lambda) \left[R(\theta_2, \lambda) + R(\theta_3, \lambda) \right] p(\lambda). \tag{28}$$

It holds that

$$L(\theta_1, \lambda) \left[R(\theta_2, \lambda) + R(\theta_3, \lambda) \right] \leq L(\theta_2, \lambda) \left[R(\theta_2, \lambda) + R(\theta_3, \lambda) \right].$$

This is easy to prove, because

$$L(\theta_2, \lambda) \left[R(\theta_2, \lambda) + R(\theta_3, \lambda) \right] = 1 + L(\theta_2, \lambda) R(\theta_3, \lambda) \tag{29}$$

This means

$$K(\theta_1, \theta_2) + K(\theta_1, \theta_3) \leq 1 + K(\theta_2, \theta_3) \tag{30}$$

2 Gedankenexperiment with spin (GHZM)

2.1 Set-up of the experiment

We now dicuss the Gedankenexperiment of GHZM [4], which shows clearly a conflict between QM and claccial local theopries.

There is one sender and three receivers, see Fig. 4.

The sender send 3 "objects": first object to receiver 1, second object to receiver 2, third object to receiver 3. Each receiver has a measuring device. He can chose to measure the property A or the property B of the object (but not both at the same time). The result of A can be $+1$ or -1; the same holds for B.

Important: the receivers receive their object at the same time. They are supposed to be far away from each other. They are also not connected in a causal way (space-like separation). So, there is no way to influence each other measurement.

Let us make a classical example of what the signals could be. The sender sends a small piece of paper to each receiver. One side is yellow, the other blue. On both sides of piece of paper he writes a number: either 1 or -1. For instance, in the first experiment he sends the following three piece of paper (the first to receiver 1, the second to receiver 2, the third to receiver 3) with the following numbers:

$$\text{1-st time: } \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \tag{31}$$

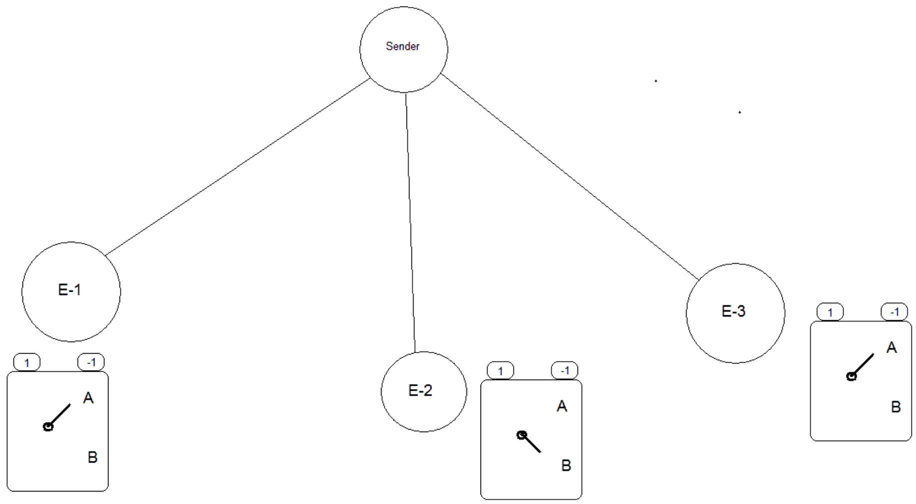

Figure 4: GHZM setup.

The property A is the number on the yellow side (upper number), the property B is the number on the blue side (lower number). The, the sender prepares three other 3 piece of papers with the following numbers

$$\text{2-nd time: } \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{32}$$

and sends them to the receivers.

The third time in a similar way:

$$\text{3-rd time: } \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{33}$$

And so on and so forth. The experiment is repeated many times. Each receiver makes a list of the chosen property (A or B) and the obtained result. After the experiment is over, they meet somewhere and prepare a table.

Table 1: Results of the experiment

1	2	3
$A_1 = +1$	$B_2 = -1$	$B_3 = -1$
$A_1 = +1$	$B_2 = -1$	$A_3 = 1$
$B_1 = +1$	$A_2 = -1$	$B_3 = -1$
...	...	...

The three experimenters agree on the following fact: every time one A and 2 B were measured, the product was +1. In formula they can summarize these results as:

$$A_1 B_2 B_3 = 1 \tag{34}$$

$$A_2 B_1 B_3 = 1 \tag{35}$$

$$A_3 B_1 B_2 = 1 \tag{36}$$

Well, they are very happy about this result and they immediately realize that they can make a further clear-cut prediction about their sender (and the way the "objects", whatever they are, are prepared). They all agree that when all three experimenters have measured A the product should be $+1$

$$A_1A_2A_3 = 1 \tag{37}$$

The proof of this expectation is very simple:

$$1 = A_1B_2B_3 \cdot A_2B_1B_3 \cdot A_3B_1B_2 = A_1A_2A_3B_1^2B_2^2B_3^2 = A_1A_2A_3. \tag{38}$$

Well, it really seems that whatever the sender is sending, this basic property must be satisfied. It was proven only out of simple logical considerations.

The three experimenters can check if the property $A_1A_2A_3 = 1$ holds or not. It is enough that they check in their table all the cases in which all three have chosen to measure A. They would actually be extremely surprised if they would, instead, find out that -contrary to their basic expectations- they find

$$A_1A_2A_3 = -1!!! \tag{39}$$

Well, if the sender decided to use electrons whose spin is prepared in a certain way this amazing and counter-intuitive result is actually realizable. And not only one time by chance, but all the time with probability 1. In order to see how this is possible a small digression on spin is needed.

2.2 Recall of the properties of the spin

We start from the basis in the z-direction: $|+\rangle$ and $|-\rangle$. We introduce the operator σ_z such that

$$\sigma_z |+\rangle = |+\rangle\,,\ \sigma_z |-\rangle = -|-\rangle\,. \tag{40}$$

(The spin operator $S_z = \frac{\hbar}{2}\sigma_z$. $|+\rangle$ represents a state with spin up in the z direction, $|-\rangle$ with spin down.)

In the given basis

$$\sigma_z = (|+\rangle\,, |-\rangle) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \langle +| \\ \langle -| \end{pmatrix}. \tag{41}$$

Now, how to express the spin in a direction which is not the z-axis?

A state with positive spin in the x-direction reads

$$|x, +\rangle = \sqrt{\frac{1}{2}}\,(|+\rangle + |-\rangle) \tag{42}$$

while the negative one

$$|x, -\rangle = \sqrt{\frac{1}{2}}\,(|+\rangle - |-\rangle)\,. \tag{43}$$

The corresponding operator σ_x is defined as

$$\sigma_x = (|+\rangle\,, |-\rangle) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \langle +| \\ \langle -| \end{pmatrix}. \tag{44}$$

In this way:

$$\sigma_x |+\rangle = |-\rangle \tag{45}$$

$$\sigma_x |-\rangle = |+\rangle \tag{46}$$

This means that σ_x interchange $|+\rangle$ with $|-\rangle$. In virtue of this property the states $|x,+\rangle$ and $|x,-\rangle$ are eigenstates of σ_x :

$$\sigma_x\,|x,+\rangle = |x,+\rangle \tag{47}$$
$$\sigma_x\,|x,-\rangle = -\,|x,-\rangle\,. \tag{48}$$

We can also invert the latter and get:

$$|+\rangle = \sqrt{\frac{1}{2}}\,(|x,+\rangle + |x,-\rangle) \tag{49}$$
$$|-\rangle = \sqrt{\frac{1}{2}}\,(|x,+\rangle - |x,-\rangle) \tag{50}$$

The spin in the positive and negative y direction is constructed as follows:

$$|y,+\rangle = \sqrt{\frac{1}{2}}\,(|+\rangle + i\,|-\rangle) \tag{51}$$

$$|y,-\rangle = \sqrt{\frac{1}{2}}\,(|+\rangle - i\,|-\rangle)\,. \tag{52}$$

Inverting:

$$|+\rangle = \sqrt{\frac{1}{2}}\,(|y,+\rangle + |y,-\rangle) \tag{53}$$
$$|-\rangle = \frac{1}{i}\sqrt{\frac{1}{2}}\,(|x,+\rangle - |x,-\rangle) \tag{54}$$

The operator σ_y is defined as

$$\sigma_y = (|+\rangle\,, |-\rangle) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \langle +| \\ \langle -| \end{pmatrix}. \tag{55}$$

That means

$$\sigma_y\,|+\rangle = -i\,|-\rangle \tag{56}$$
$$\sigma_x\,|-\rangle = i\,|+\rangle \tag{57}$$

In this way the states $|y,+\rangle$ and $|y,-\rangle$ are eigenstates of σ_y.

$$\sigma_y\,|y,+\rangle = |y,+\rangle \tag{58}$$
$$\sigma_y\,|y,-\rangle = -\,|y,-\rangle\,. \tag{59}$$

2.3 Sender sending electrons

The sender is not sending classical objects, but is sending three electrons. The three electrons are in the following entangled quantum state:

$$|S\rangle = \sqrt{\frac{1}{2}}\,(|+++\rangle - |---\rangle) \tag{60}$$

Now, the measurements A and B correspond indeed to σ_x and σ_y;

$$A \equiv \sigma_x, B \equiv \sigma_y. \tag{61}$$

In the case in which the first receiver measure A and the other 2 B we have the following property:

$$\sigma_x^{(1)}\sigma_y^{(2)}\sigma_y^{(3)} |S\rangle = |S\rangle \, . \tag{62}$$

$|S\rangle$ is an eigenstate of this operator. That means, the product $A_1 B_2 B_3 = 1$. Always, with no exceptions. Similarly:

$$\sigma_y^{(1)}\sigma_x^{(2)}\sigma_y^{(3)} |S\rangle = |S\rangle \tag{63}$$

$$\sigma_y^{(1)}\sigma_y^{(2)}\sigma_x^{(3)} |S\rangle = |S\rangle \, . \tag{64}$$

This means:

$$A_2 B_1 B_3 = 1 \tag{65}$$

$$A_3 B_1 B_2 = 1 \tag{66}$$

But now let us turn to the cases in which all 3 receivers decide to measure A. Also in this case $|S\rangle$ is an eigenstate, but:

$$\sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)} |S\rangle = -|S\rangle \; ! \tag{67}$$

That is:

$$A_1 A_2 A_3 = -1 \, . \tag{68}$$

The classically completely unexpected result has appeared.

Note, just as in the Bell case, no classical local real theory can generate this result. But, there is an advantage in this case: the classical and the quantum cases predict the opposite results. $A_1 A_2 A_3 = +1$ classically, $A_1 A_2 A_3 = -1$ quantistically. Utterly opposite. No way to make them coincide. This example is also pretty easy to remember.

(How to calculate these results explicitly? One can of course also perform an explicit calculation. In the case when ABB is measured, we have to rewrite the first electron in the $|x,+\rangle$ $|x,-\rangle$ basis and the second and the third electron in the $|y,+\rangle$ and $|y,-\rangle$ basis. One can then see that in all possible combinations the result $A_1 B_2 B_3 = 1$ holds, as it must because the state is an eigenstate. The same for the other cases. It is a bit tedious but straightforward calculation.)

References

References

[1] Z.K. Silagadze, "Schrodinger's cat versus Darwin" (Novosibirsk, IYF & Novosibirsk State U.) . Mar 2009. 52pp. e-Print: arXiv:0903.5539 [physics.soc-ph]

[2] A. Zeilinger, Einsteins Spuck, Goldmann Verlag

[3] J.S. Bell: Speakable and Unspeakable in Quantum Mechanics (Cambridge University Press 1987).

[4] http://en.wikipedia.org/wiki/GHZ_experiment. See also the wunderful video of a lecture of Sidney Coleman: http://media.physics.harvard.edu/video/index.php?id=SidneyColeman_QMIYF.flv