The "Dome": an unexpectedly simple failure of determinism.

by J. D. Norton.

Force:

$$F(x) = c\sqrt{x}$$

The e.o.m. reads:

$$m\ddot{x} = F = c\sqrt{x}$$

Let us consider the following simple initial condition.

$$\begin{cases} x(0) = 0 \\ \dot{x}(0) = v_0 = 0 \end{cases}$$

It is then clear that $\boxed{x(t) = 0}$ $\forall t$ is a solution of the e.o.m. with the correct initial conditions.

One would 'naively' expect that this solution is unique!

Is this the case here? $\rightsquigarrow$ NO!

Let us search a solution of the form:

$$x(t) = \alpha t^4 \qquad \alpha > 0 .$$

$$\dot{x} = 4\alpha t^3$$

$$\ddot{x} = 12\alpha t^2$$

$$m\ddot{x} = c\sqrt{x}$$

$$\Downarrow$$

$$m \cdot 12\alpha t^2 = c\sqrt{\alpha t^4}$$

Ergo:

$$12\, m\alpha = c\sqrt{\alpha} \quad \rightarrow \quad \sqrt{\alpha} = \frac{c}{12m} \qquad \alpha = \frac{c^2}{14m^2} > 0$$

We then have a second solution, $x = \alpha t^4$, which also fullfills the initial conditions $x(0) = \dot{x}(0) = 0$!

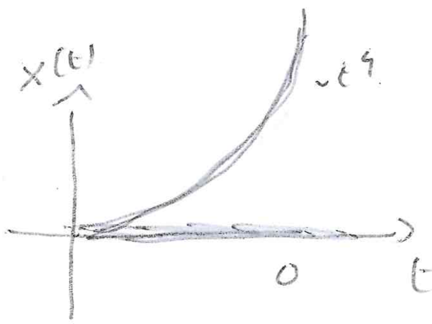
Then, such system allows for 2 solutions...

which one will be picked up in a "real experiment"?

Two solutions:

$x_1(t) = 0$

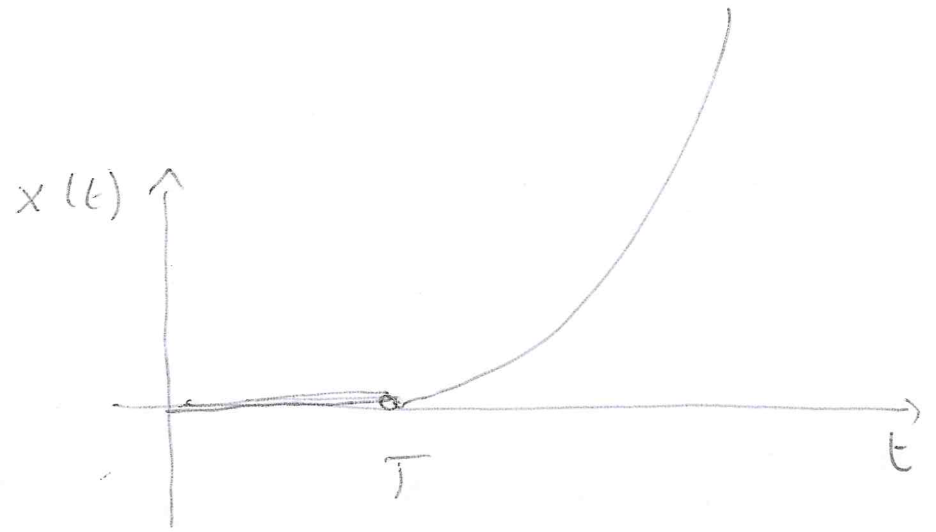
$x_2(t) = \alpha t^4$



Even worse; the following trajectory

$$x(t) = \begin{cases} 0 & t \leq T \\ \alpha (t-T)^4 & t > T \end{cases}$$

is also a solution!



The particle "waits" for a certain time interval T in the origin, then all of a sudden it starts.

T is not determined... it is arbitrary.

That means that we have an ∞-no of solutions

$$x_T(t) = \begin{cases} 0 & t \leq T \\ \alpha (t-T)^4 & t > T \end{cases} \qquad \forall T \ldots$$