Computational Methods for Kinetic Processes in Plasma Physics

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Basics of PIC Simulation methods

* Collisionless plasmas
* Finite-size particles
* Electrostatic codes
* Charge assignment and force interpolation (already in 3-D system)
* Filtering action of shape function
* Summary

June 1, 2015

Context

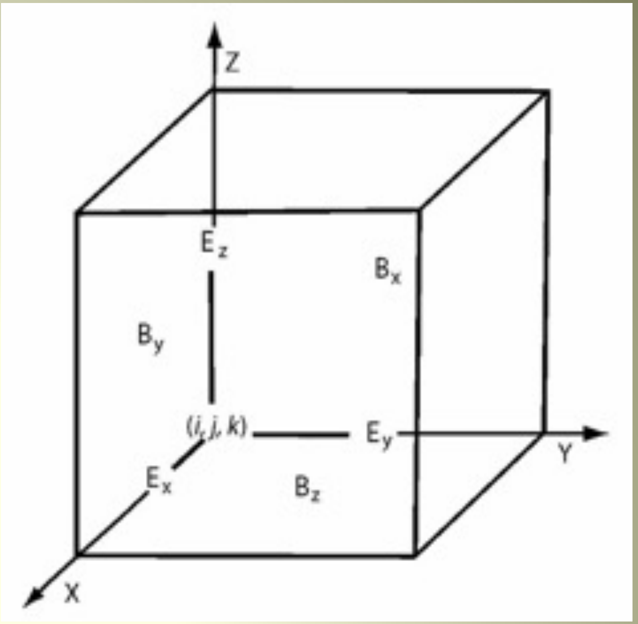
- Three-dimensional current deposit
  by Villasenor & Buneman
- Zigzag scheme in two-dimensional systems
  by Umeda

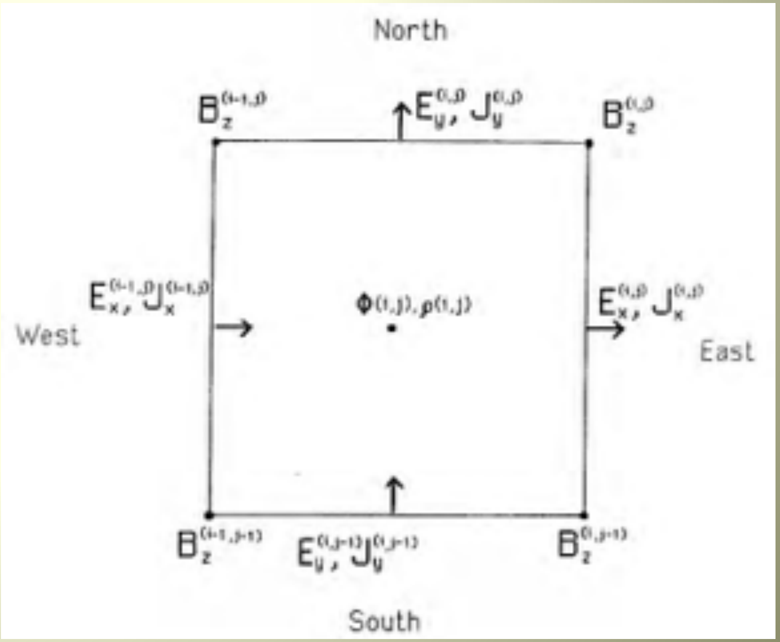
Staggered mesh with E and B

$$B_z^{new} - B_z^{old} = \frac{\delta t}{\delta x}\left((E_x^{east} - E_x^{west}) - (E_y^{north} - E_y^{south})\right),$$

$$E_x^{new} - E_x^{old} = \frac{\delta t}{\delta x}(B_z^{north} - B_z^{south}) - \delta t J_x,$$

$$E_y^{new} - E_y^{old} = \frac{\delta t}{\delta x}(B_z^{east} - B_z^{west}) - \delta t J_y$$





Current deposit scheme (2-D)

$$\nabla \cdot E = 4\pi\rho, \quad \nabla \cdot \frac{\partial E}{\partial t} = 4\pi \frac{\partial \rho}{\partial t}, \quad \nabla \cdot (c\nabla \times B - 4\pi J) = 4\pi \frac{\partial \rho}{\partial t},$$

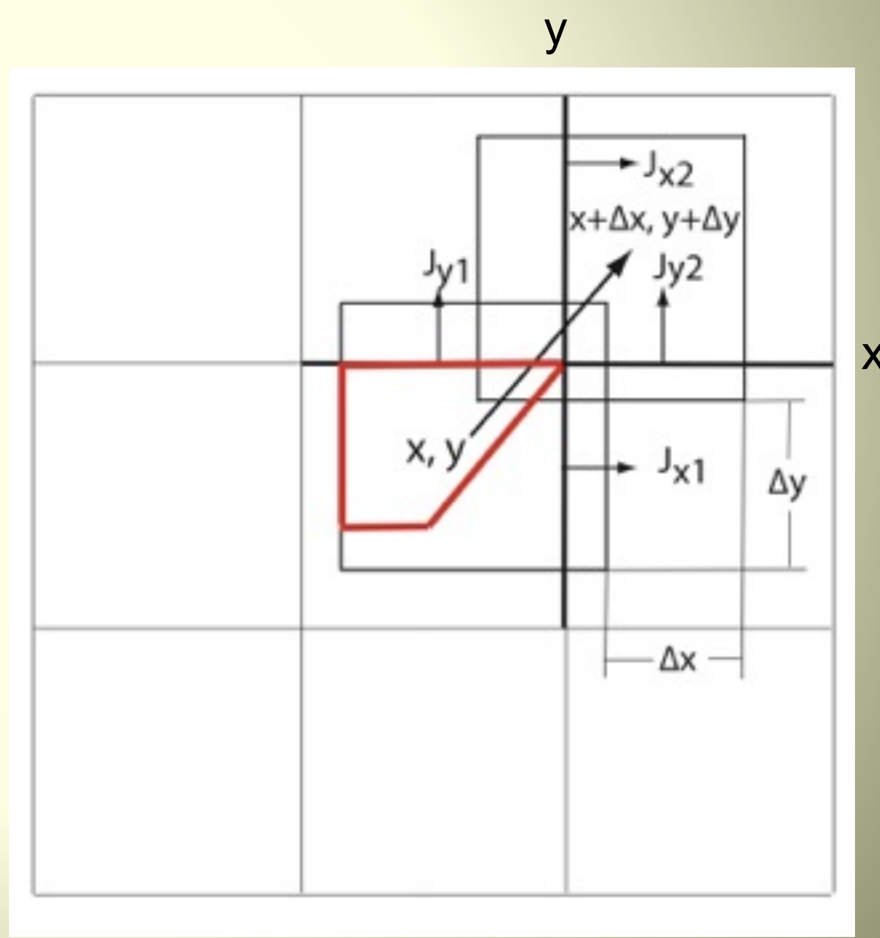
$$\frac{\partial \rho}{\partial t} = -\nabla \cdot J$$

$$J_{x1} = q\Delta x\left(\frac{1}{2} - y - \frac{1}{2}\Delta y\right)$$

$$J_{x2} = q\Delta x\left(\frac{1}{2} + y + \frac{1}{2}\Delta y\right)$$

$$\rightarrow \quad J_{y1} = q\Delta y\left(\frac{1}{2} - x - \frac{1}{2}\Delta x\right)$$

$$J_{y2} = q\Delta y\left(\frac{1}{2} + x + \frac{1}{2}\Delta x\right)$$



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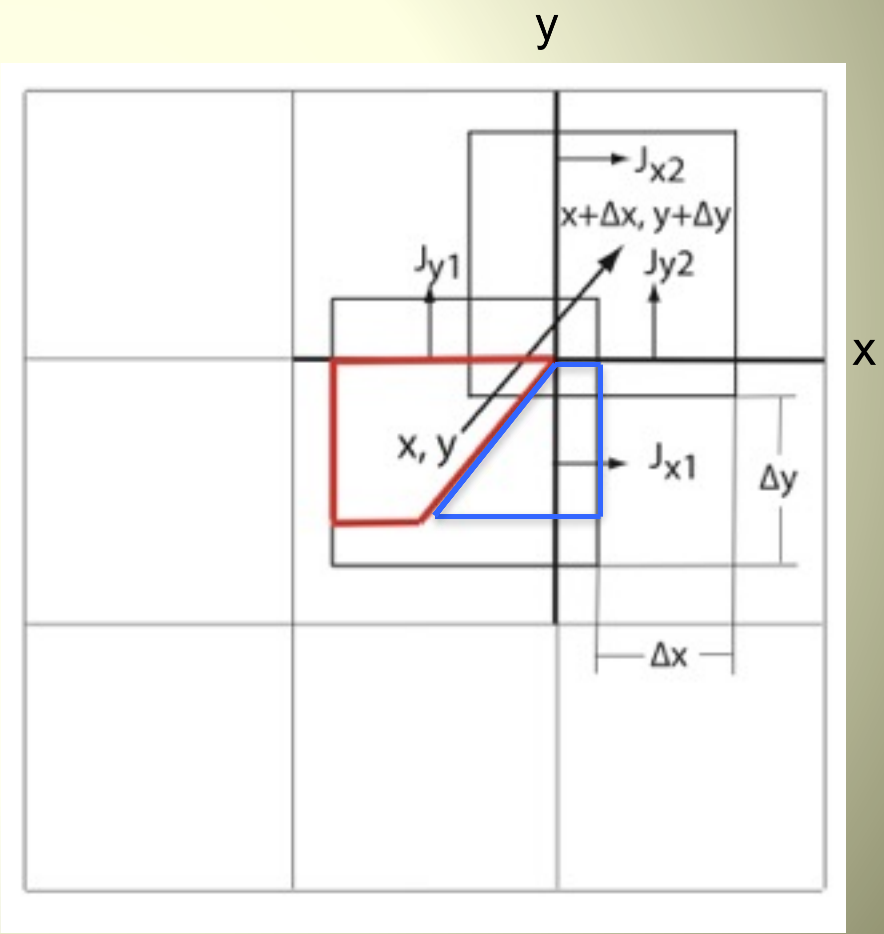
$$J_{y1} = q\Delta y\left(\frac{1}{2} - x - \frac{1}{2}\Delta x\right)$$

$$J_{y2} = q\Delta y\left(\frac{1}{2} + x + \frac{1}{2}\Delta x\right)$$



3D current deposit

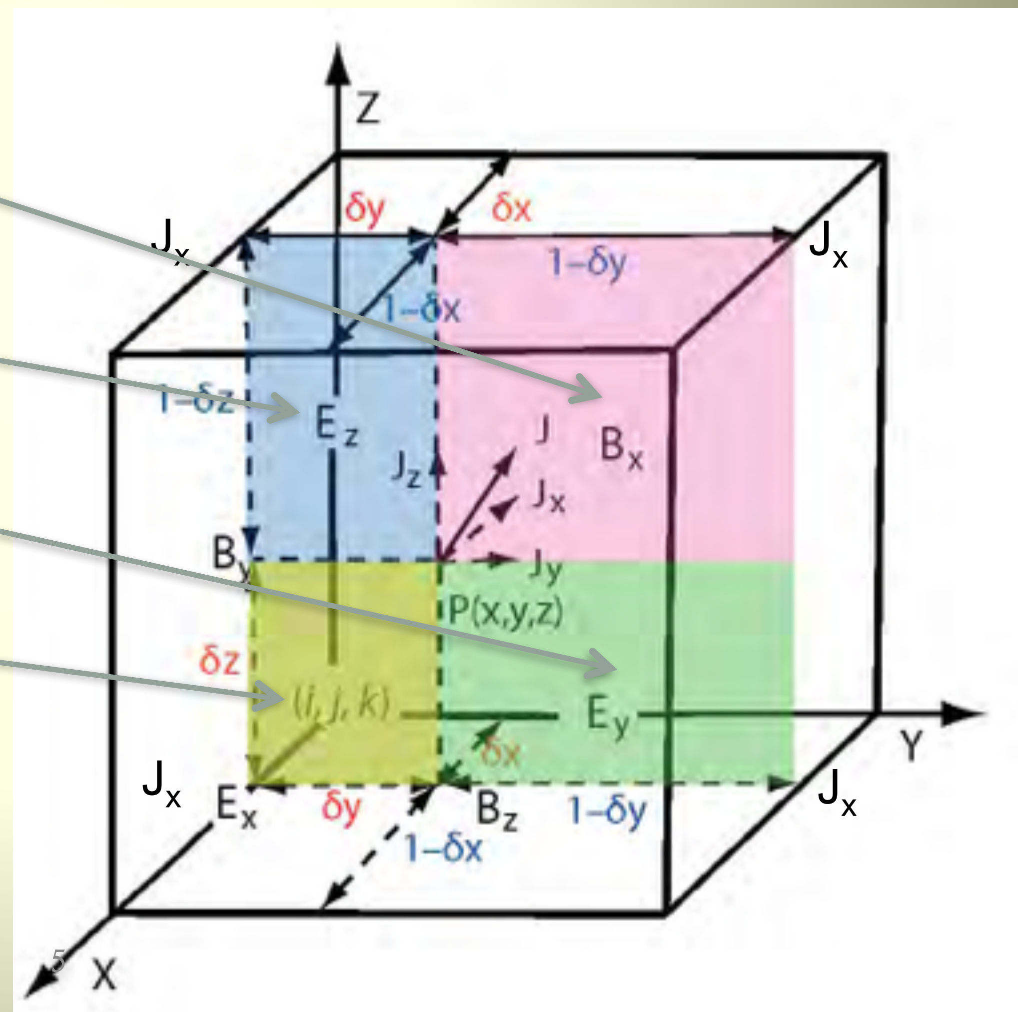
$$x = i + \delta x = i + \xi,\ y = j + \delta y = j + \eta,\ z = k + \delta z = k + \zeta$$

$$(1-\eta)(1-\zeta)$$

$$\eta(1-\zeta)$$

$$(1-\eta)\zeta$$

$$\eta\zeta$$



3D current deposit

Particle moves from $(i+\xi_1, j+\eta_1, k+\zeta_1)$ to $(i+\xi_2, j+\eta_2, k+\zeta_2)$

$\Delta x = \xi_2 - \xi_1,\ \Delta y = \eta_2 - \eta_1,\ \Delta z = \zeta_2 - \zeta_1$ between t= −1/2 and t = +1/2

at t = 0 $\bar{\zeta} = (\xi_2 + \xi_1)/2, \bar{\eta} = (\eta_2 + \eta_1)/2, \bar{\zeta} = (\zeta_2 + \zeta_1)/2$

- $J_x = \int_{\xi_1}^{\xi_2} \eta(t)\zeta(t)\,d\xi$

$$= \int_{-1/2}^{1/2} (\bar{\eta} + t\Delta y)(\bar{\zeta} + t\Delta z)\Delta x\,dt$$

$$= \Delta x \bar{\eta}\bar{\zeta} + \Delta x \Delta y \Delta z / 12$$

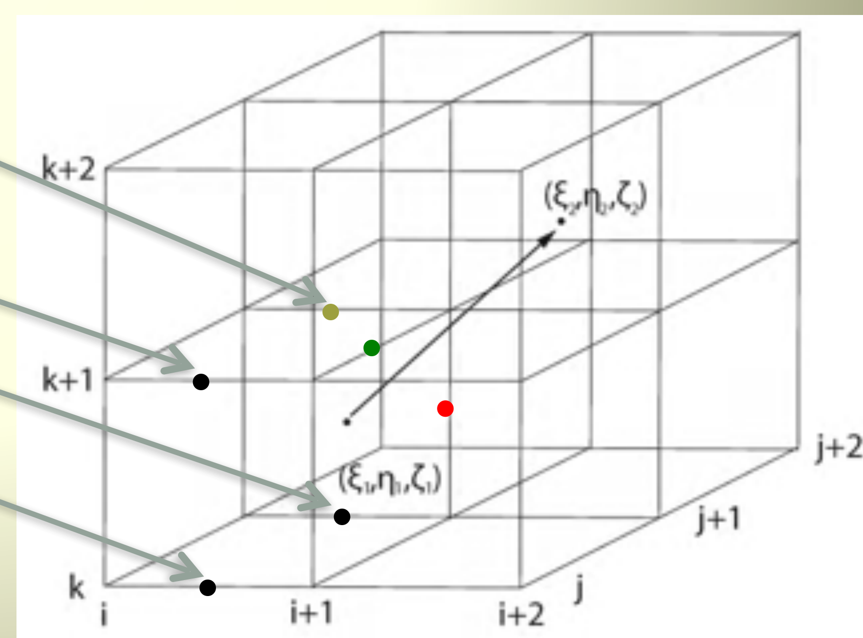
$$\Delta x(1-\bar{\eta})\bar{\zeta} - \Delta x \Delta y \Delta z / 12$$

$$\Delta x \bar{\eta}(1-\bar{\zeta}) - \Delta x \Delta y \Delta z / 12$$

$$\Delta x(1-\bar{\eta})(1-\bar{\zeta}) + \Delta x \Delta y \Delta z / 12$$

$$i, \Delta x, \bar{\eta} \Rightarrow j, \Delta y, \bar{\xi} \Rightarrow k, \Delta z, \bar{\zeta} \Rightarrow i, \Delta x, \bar{\eta}$$

- J_y at (i+1, j+1/2, k+1)
- J_z at (i+1, j+1, k+1/2)



The total fluxes into the cell indexed i+1, j+1, k+1

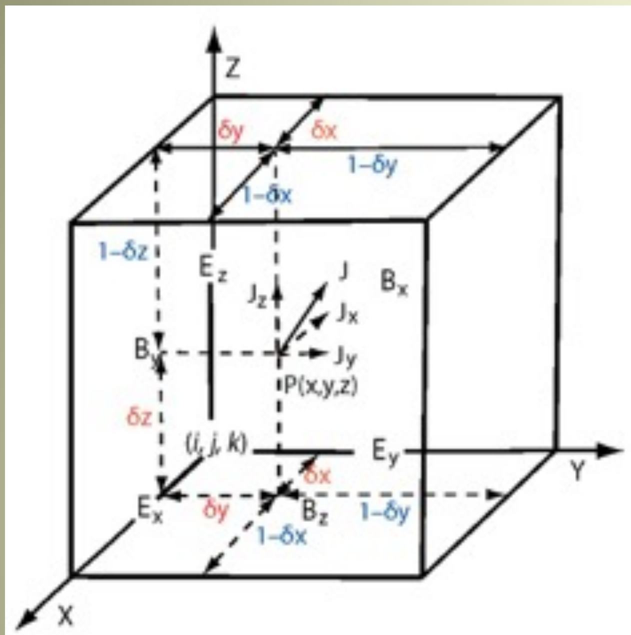
$$\Delta x \bar{\eta}\bar{\zeta} + \Delta y \bar{\zeta}\bar{\xi} + \Delta z \bar{\xi}\bar{\eta} + \Delta x \Delta y \Delta z / 4$$

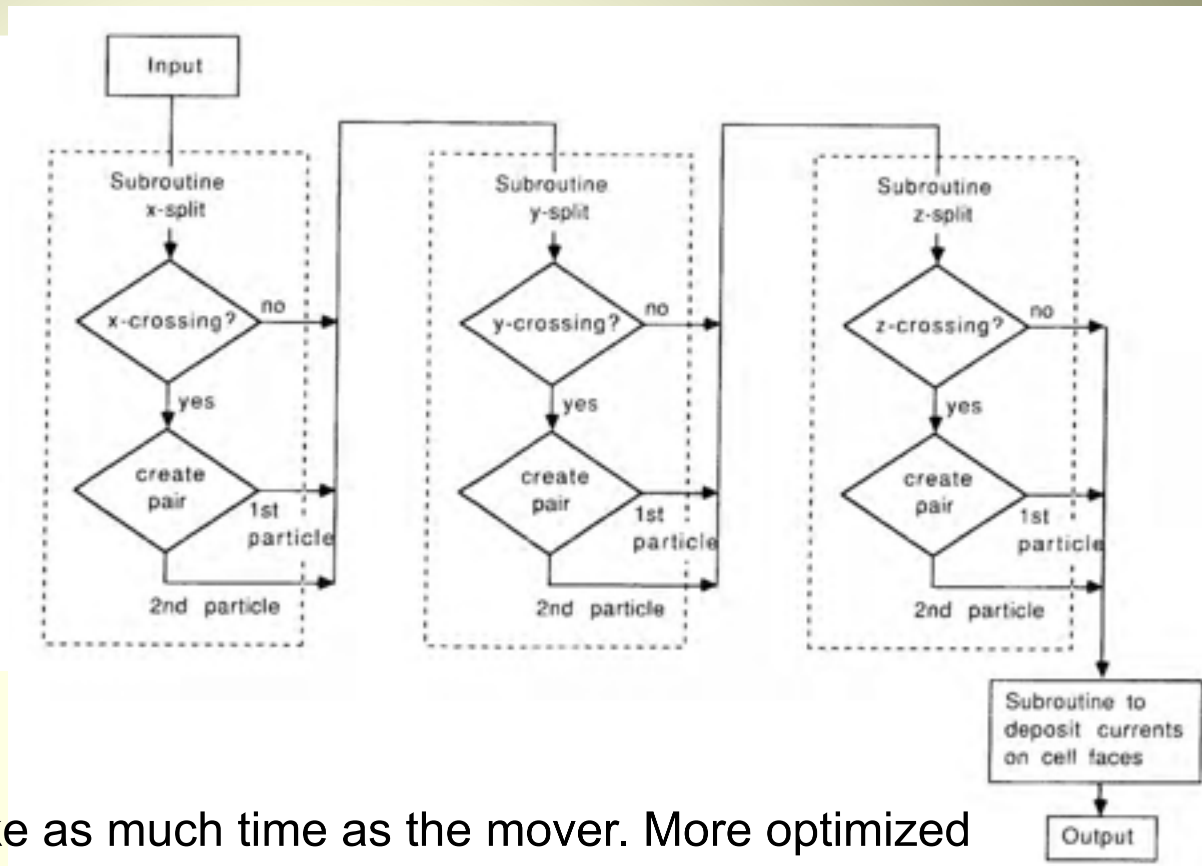
$$= (\bar{\xi} + \tfrac{1}{2}\Delta x)(\bar{\eta} + \tfrac{1}{2}\Delta y)(\bar{\zeta} + \tfrac{1}{2}\Delta z)$$

$$- (\bar{\xi} - \tfrac{1}{2}\Delta x)(\bar{\eta} - \tfrac{1}{2}\Delta y)(\bar{\zeta} - \tfrac{1}{2}\Delta z)$$

The difference between the particle fractions protruding into the cell before and after the move.

Charge and current deposition

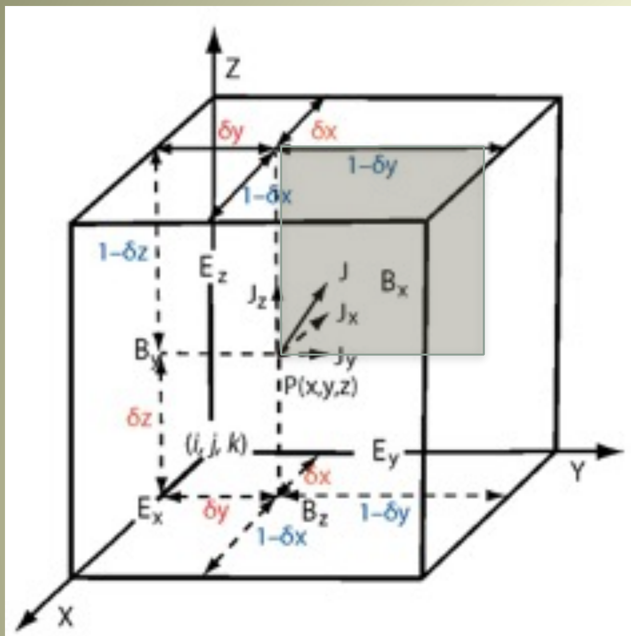


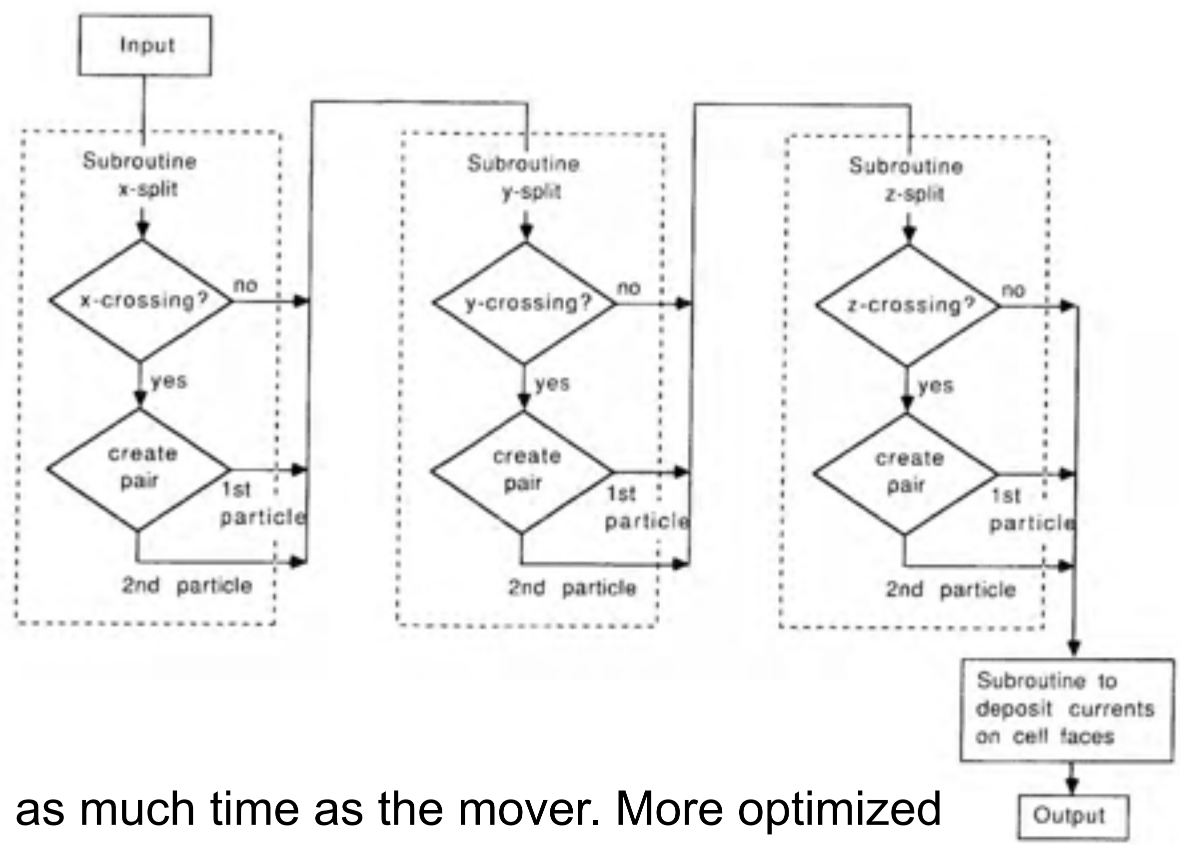


Current deposition can take as much time as the mover. More optimized deposits exist (Umeda 2003).

Charge conservation makes the whole Maxwell solver local and hyperbolic. Static fields can be established dynamically.

Charge and current deposition





Current deposition can take as much time as the mover. More optimized deposits exist (Umeda 2003).

Charge conservation makes the whole Maxwell solver local and hyperbolic. Static fields can be established dynamically.

```
c  BOUNDARY CONDITIONS FOR PARTICLES AND CURRENT DEPOSITION
c  current smoothing (filtering) is applied here after current deposition
c  from all particles; this saves computational time because particle number
c  is much larger than grid points number and particles are uniformly distributed
c  increaments to the electric fields from current deposition are stored in
c  "dex, dey, dez" arrays as referenced now by "Split_..." subroutines, and
c  by "xsplit,..., depsit", "Split_..." updated for boundary conditions handling
c ** there is no need   for "Clear_ghost" routine now **

c  ambient ions and electrons (periodicity in y- and z-direction applied)
      call Split_Ambient(ions,xi,yi,zi,ui,vi,wi,mb,dex,dey,dez,
     &                 mFx,mFy,mFz,mx,my,mz,qi,DHDx,DHDy,DHDz,
     &                 PVLeft,PVRght,PBFrnt,PBRear,PBBot,PBTop,
     &                   vithml,c,DT,refli,elosi,mi,sele,0,is,
     &                              ijittne,njitt,njitte0)
```

```
     call Split_Ambient(lecs,xe,ye,ze,ue,ve,we,mb,dex,dey,dez,
    &                 mFx,mFy,mFz,mx,my,mz,qe,DHDx,DHDy,DHDz,
    &                 PVLeft,PVRght,PBFrnt,PBRear,PBBot,PBTop,
    &                 vethml,c,DT,refle,elose,me,sele,isel,is,
    &                                ijittne,njitt,njitte0)
c  jet ions and electrons (periodicity in y- and z-direction applied)
      if (ionj.ne.0) then
c      write(6,*)'mFz=',mFz
       call Split_JET(ionj,xij,yij,zij,uij,vij,wij,mj,dex,dey,dez,
    &                  mFx,mFy,mFz,my,mz,qi,DHDx,DHDy,DHDz,DT)
      end if

      if (lecj.ne.0) then
c      write(6,*)'mFz=',mFz
       call Split_JET(lecj,xej,yej,zej,uej,vej,wej,mj,dex,dey,dez,
    &                  mFx,mFy,mFz,my,mz,qe,DHDx,DHDy,DHDz,DT)
      end if

c  pass contributions from current deposit to E-fields from ghost cells
c  to the cells in the appropriate domain's "particle core" before E-field
c  passing and applying periodic boundary conditions
c ** "addlayer" subroutine is embedded in here **
c !!! subroutine must be called with "FBDRxe" and "FBDLxe" !!!
```

```
c *************************************************************
     subroutine Split_JET(ipar,x,y,z,u,v,w,mh,dex,dey,dez,
    &                    mFx,mFy,mFz,my,mz,q,DHDx,DHDy,DHDz,dt)

     dimension dex(mFx,mFy,mFz),dey(mFx,mFy,mFz),dez(mFx,mFy,mFz)
     dimension x(mh),y(mh),z(mh)
     dimension u(mh),v(mh),w(mh)

     k=0
50   k=k+1
c  previous particle position
     x0=x(k)-u(k)*dt
     y0=y(k)-v(k)*dt
     z0=z(k)-w(k)*dt

c  make particles which are out of the box (3,my-2)*(3,mz-2)
c  periodic in transverse dimensions with period my-5 (mz-5)
c ** in 3D version particles that were out of the virtual box transverse    **
c ** boundaries have been already passed to processes according to the      **
c ** periodic boundary conditions; here only their proper y and z-positions **
c ** in the new domain are calculated (e.g. y(k)=83.5 --> y(k)=3.5 (my=85)) **
cJET only periodicity applied
     per = sign(.5*(my-5.),y(k)-3.) + sign(.5*(my-5.),y(k)-my+2.)
     y(k) = y(k)-per
     y0   = y0  -per
```

```
      per = sign(.5*(mz-5.),z(k)-3.) + sign(.5*(mz-5.),z(k)-mz+2.)
      z(k) = z(k)-per
      z0   = z0  -per

cU1      call depsitUM2(x(k),y(k),z(k),x0,y0,z0,dex,dey,dez,mFx,mFy,mFz,
cU1     &                                          q,DHDx,DHDy,DHDz)
      call depsitUM1(x(k),y(k),z(k),x0,y0,z0,dex,dey,dez,mFx,mFy,mFz,
     &                                          q,DHDx,DHDy,DHDz)

53    if (k.lt.ipar) goto 50

      return
      end
```

```
c ************************************************************
      subroutine Split_Ambient(ipar,x,y,z,u,v,w,mh,dex,dey,dez,
     &             mFx,mFy,mFz,mx,my,mz,q,DHDx,DHDy,DHDz,
     &            PVLeft,PVRght,PBFrnt,PBRear,PBBot,PBTop,
cNewJacek2
     &             vthml,c,dt,refl,eloss,mass,s,isel,is,
cNewJacek3
     &                        ijittn,njitt,njitt0)
c    &             vthml,c,dt,refl,eloss,mass,is)

      real mass

      dimension dex(mFx,mFy,mFz),dey(mFx,mFy,mFz),dez(mFx,mFy,mFz)
      dimension x(mh),y(mh),z(mh)
      dimension u(mh),v(mh),w(mh)
cNewJacek2
      integer isel
      integer s(mh)
cNewJacek3
      integer ijittn(njitt)

      pi  = 4*atan(1.0d0)

c  variance of Maxwell distribution for individual velocity components
      sig = vthml/sqrt(2.)
```

```
      k=0
50    k=k+1
c  previous particle position needed for current deposition
      x0=x(k)-u(k)*dt
      y0=y(k)-v(k)*dt
      z0=z(k)-w(k)*dt

      xini=x(k)
      reflect=0.0

c  particles outside the box (3,mx-2) are reflected or eliminated from the
c  simulations
c  !!! now the boundaries are (PVLeft,PVRght) !!!
c ** only coords(1)=0 and coords(1)=Npx-1 processors calculate this part **
c       if (x(k).ge.3. .and. x(k).lt.mx-2.) goto 51
      if (x(k).ge.PVLeft .and. x(k).lt.PVRght) goto 51

c  which boundary has been crossed?
c        x(k)=amin1(amax1(x(k),3.),mx-2.000001)
       x(k)=amin1(amax1(x(k),PVLeft),PVRght-0.000001)
c  time at boundary crossing
       tx=(x(k)-x0)/u(k)
```

```
c  y,z at boundary crossing
c ** particles which are not reflected are stopped at the boundary at this **
c ** point and deposit current at this x-position                            **
      y(k)=y0+tx*v(k)
      z(k)=z0+tx*w(k)

c4push
      reflect=refl

c ** particle reflection according to the reflection rate "refl" **
c ** e.g., refl=0.8 -> ~20% particles reflected                  **
      rabs=ran1(is)

c  particles are reflected from stiff-wall x-boundary
c  now particles reflected have new set of inward velocities, as if a new
c  particle entered the box in place of the one which left the box
c ** this can cause numerical errors in a rare case in which a reflected     **
c ** particle crosses at the same time one of the other boundaries (y or z); **
c ** such a particle has been already passed to a new processor in          **
c ** "particle_passing" and its new location after a move with new velocity  **
c ** components must fit the domain boundaries after the periodic conditions **
c ** have been applied; because of that a loop in label "20" is necessary -  **
c ** this is a change compared with older version of this subroutine: note   **
c ** that the problem does not exist when reflection from stiff-wall boundary**
c ** is applied                                                              **
```

```
      ntry = 0
      u0=u(k)
      v0=v(k)
      w0=w(k)

      if(rabs.gt.reflect) then
c       u(k)=-u(k)
c       ut=u(k)
       ut=u0
20      yt=y(k)
       zt=z(k)
       y0t=y0
       z0t=z0
       ntry = ntry + 1
c ** stiff-wall boundary reflection is applied if loop 20 does not succeed after **
c ** 1000 trials; this in particular takes care of cases in which a particle     **
c ** crosses domain x-boundaries at a very oblique angle, so that y or z position**
c ** at boundary crossing is too much outside the domain boundaries and a        **
c ** selection of loop 20 must tune up to minus original velocity to place       **
c ** particle in right domain                                                     **
      if (ntry.ge.1) then
        u(k)=-u0
        v(k)= v0
        w(k)= w0
        goto 21
       end if
```

```
18     r11 = ran1(is)
       if(r11.EQ.1.0) goto 18

       r1 = sqrt(-2.0*log(1.0-r11))
       if(sig*r1.ge.c) goto 18
       r2 = 2.0*pi*ran1(is)
       unew = sig*r1*cos(r2)
       vnew = sig*r1*sin(r2)

19     r33 = ran1(is)
       if(r33.EQ.1.0) goto 19

       r3 = sqrt(-2.0*log(1.0-r33))
       if(sig*r3.ge.c) goto 19
       r4 = 2.0*pi*ran1(is)
       wnew = sig*r3*cos(r4)

       u(k) = -sign(unew,ut)
       v(k) = vnew
       w(k) = wnew

21     yt=y(k)+(dt-tx)*v(k)
       zt=z(k)+(dt-tx)*w(k)
```

```
c apply the periodicity here and check the domain boundaries
        per = sign(.5*(my-5.),yt-3.) + sign(.5*(my-5.),yt-my+2.)
        yt = yt-per
        y0t= y0t -per

        per = sign(.5*(mz-5.),zt-3.) + sign(.5*(mz-5.),zt-mz+2.)
        zt = zt-per
        z0t= z0t -per

        if((yt.lt.PBFrnt) .or. (yt.ge.PBRear) .or.
     &     (zt.lt.PBBot ) .or. (zt.ge.PBTop ) .or.
     &     (u(k)**2+v(k)**2+w(k)**2 .ge. c**2)) goto 20

         x(k)=x(k)+(dt-tx)*u(k)
         y(k)=yt
         z(k)=zt
         y0=y0t
         z0=z0t
       endif
       go to 52

51     continue
      rabs=2.0
c52    continue
```

```
c  make particles which are out of the box (3,my-2)*(3,mz-2)
c  periodic in transversal dimensions with period my-5 (mz-5)
c ** in 3D version particles that are out of the virtual box transversal    **
c ** boundaries are passed to processes according to the periodic boundary  **
c ** conditions; here only their proper y and z-positions in the new domain **
c ** are calculated                                                          **
      per = sign(.5*(my-5.),y(k)-3.) + sign(.5*(my-5.),y(k)-my+2.)
      y(k) = y(k)-per
      y0   = y0  -per

      per = sign(.5*(mz-5.),z(k)-3.) + sign(.5*(mz-5.),z(k)-mz+2.)
      z(k) = z(k)-per
      z0   = z0  -per

c  split particles which cross cell boundaries and deposit currents
c ** Umeda's 2nd-order method **
cU1      call depsitUM2(x(k),y(k),z(k),x0,y0,z0,dex,dey,dez,mFx,mFy,mFz,
cU1     &                                      q,DHDx,DHDy,DHDz)
52    call depsitUM1(x(k),y(k),z(k),x0,y0,z0,dex,dey,dez,mFx,mFy,mFz,
     &                                      q,DHDx,DHDy,DHDz)
```

```
c  particles (in left or right processors) which are outside virtual box
c  x-boundaries (non-reflected particles) are eliminated and replaced by
c  particles from the top of the stack
c4push      if (rabs.ge.refl) goto 53
      if (rabs.gt.reflect .and. xini.ge.PVLeft) goto 53
c  checking kinetic energy lost
      eloss=eloss+0.5*mass*(u(k)**2+v(k)**2+w(k)**2)

      x(k)=x(ipar)
      y(k)=y(ipar)
      z(k)=z(ipar)
      u(k)=u(ipar)
      v(k)=v(ipar)
      w(k)=w(ipar)
cNewJacek2
      if (isel.eq.1) then
cNewJacek3
c  if eliminated particle was traced its "ijittn(i)" points the nonexisting
c  element; it will be subsequently taken out together with the radiation data
c !! this is a waste of computing time -> selection method must avoid !!
c !! choosing particles which are close to boundaries                 !!
```

```
if (s(k).eq.1) then
        do i = 1,njitt0
         if (ijittn(i).eq.k) then
          ijittn(i)=0
          goto 531
         end if
        end do
531     Continue
       end if
c replace with new particle here
       s(k)=s(ipar)
       if (s(k).eq.1) then
        do i = njitt0,1,-1
         if (ijittn(i).eq.ipar) then
          ijittn(i)=k
          goto 532
         end if
        end do
532     Continue
       end if
      end if
      ipar=ipar-1
      k=k-1
53    if (k.lt.ipar) goto 50
      return
      end
```

Zigzag scheme in two-dimensional systems

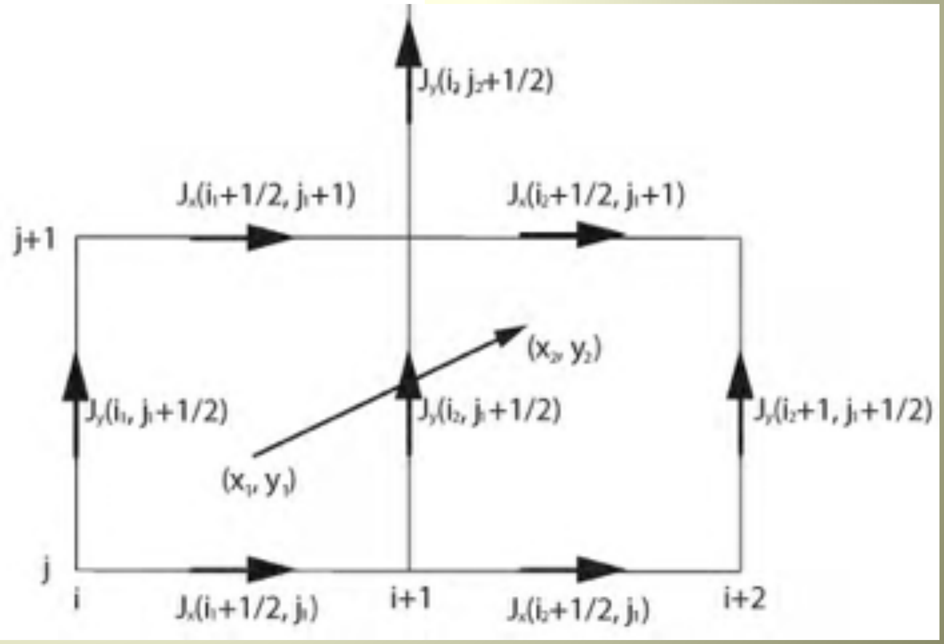
$$\frac{J_x^{t+\Delta t/2}(i+\frac{1}{2},j) - J_x^{t+\Delta t/2}(i-\frac{1}{2},j)}{\Delta x} + \frac{J_y^{t+\frac{\Delta t}{2}}(i,j+\frac{1}{2}) - J_y^{t+\Delta t/2}(i,j-\frac{1}{2})}{\Delta y}$$
$$= \frac{\rho^t(i,j) - \rho^{t+\Delta t}(i,j)}{\Delta t}.$$

$$J_x\left(i_1+\tfrac{1}{2}, j_1\right) = \frac{1}{\Delta x \Delta y} F_x(1-W_y), \qquad J_x\left(i_1+\tfrac{1}{2}, j_1+1\right) = \frac{1}{\Delta x \Delta y} F_x W_y,$$
$$J_y\left(i_1, j_1+\tfrac{1}{2}\right) = \frac{1}{\Delta x \Delta y} F_y(1-W_x), \qquad J_y\left(i_1+1, j_1+\tfrac{1}{2}\right) = \frac{1}{\Delta x \Delta y} F_y W_x,$$

$$i_1 \equiv \text{floor}(x_1/\Delta x), \qquad i_2 \equiv \text{floor}(x_2/\Delta x),$$
$$j_1 \equiv \text{floor}(y_1/\Delta y), \qquad j_2 \equiv \text{floor}(y_2/\Delta y),$$

$$F_x = q\frac{x_2 - x_1}{\Delta t}, \qquad F_y = q\frac{y_2 - y_1}{\Delta t},$$

$$W_x = \frac{x_1 + x_2}{2\Delta x} - i_1, \qquad W_y = \frac{y_1 + y_2}{2\Delta y} - j_1.$$



see Umeda (2003) for detailed numerical method

Zigzag scheme in two-dimensional systems

$$\frac{J_x^{t+\Delta t/2}(i+\frac{1}{2},j) - J_x^{t+\Delta t/2}(i-\frac{1}{2},j)}{\Delta x} + \frac{J_y^{t+\frac{\Delta t}{2}}(i,j+\frac{1}{2}) - J_y^{t+\Delta t/2}(i,j-\frac{1}{2})}{\Delta y} = \frac{\rho^t(i,j) - \rho^{t+\Delta t}(i,j)}{\Delta t}.$$

$$J_x\left(i_1+\tfrac{1}{2},j_1\right) = \frac{1}{\Delta x \Delta y} F_x(1-W_y), \qquad J_x\left(i_1+\tfrac{1}{2},j_1+1\right) = \frac{1}{\Delta x \Delta y} F_x W_y,$$

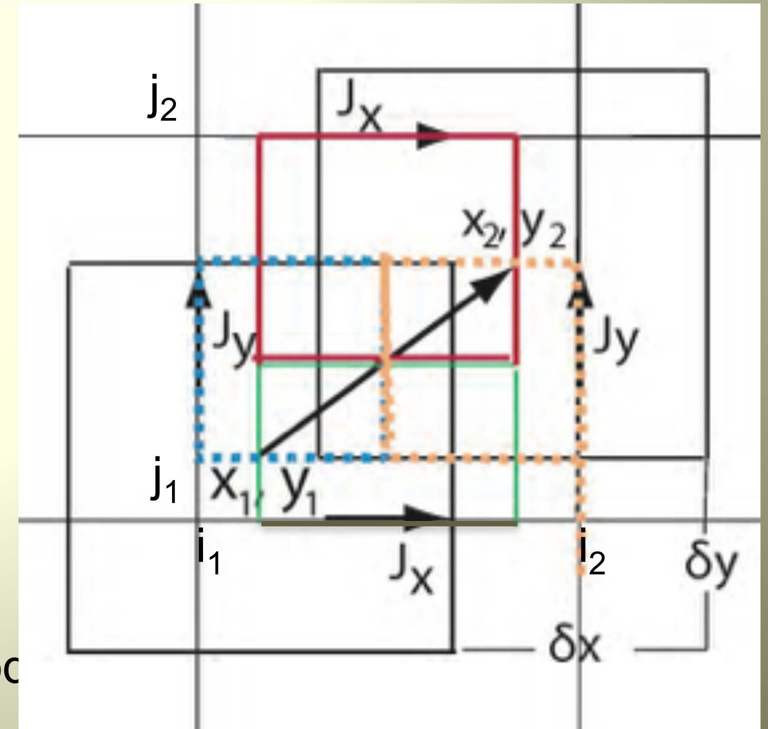
$$J_y\left(i_1,j_1+\tfrac{1}{2}\right) = \frac{1}{\Delta x \Delta y} F_y(1-W_x), \qquad J_y\left(i_1+1,j_1+\tfrac{1}{2}\right) = \frac{1}{\Delta x \Delta y} F_y W_x,$$

$i_1+1=i_2$ and $j_1+1=j_2$

$$i_1 \equiv \mathrm{floor}(x_1/\Delta x), \qquad i_2 \equiv \mathrm{floor}(x_2/\Delta x),$$
$$j_1 \equiv \mathrm{floor}(y_1/\Delta y), \qquad j_2 \equiv \mathrm{floor}(y_2/\Delta y),$$

$$F_x = q\frac{x_2-x_1}{\Delta t}, \qquad F_y = q\frac{y_2-y_1}{\Delta t},$$

$$W_x = \frac{x_1+x_2}{2\Delta x} - i_1, \qquad W_y = \frac{y_1+y_2}{2\Delta y} - j_1.$$



see Umeda (2003) for detailed numerical method

Zigzag scheme in two-dimensional systems

$$\frac{J_x^{t+\Delta t/2}(i+\frac{1}{2},j)-J_x^{t+\Delta t/2}(i-\frac{1}{2},j)}{\Delta x}+\frac{J_y^{t+\frac{\Delta t}{2}}(i,j+\frac{1}{2})-J_y^{t+\Delta t/2}(i,j-\frac{1}{2})}{\Delta y}$$
$$=\frac{\rho^t(i,j)-\rho^{t+\Delta t}(i,j)}{\Delta t}.$$

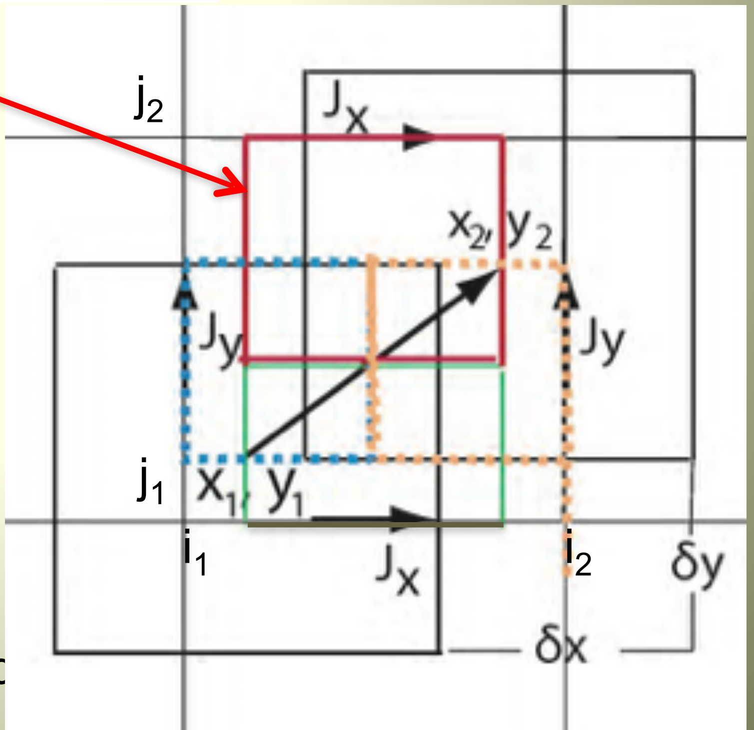
$$J_x\left(i_1+\tfrac{1}{2},j_1\right)=\frac{1}{\Delta x\Delta y}F_x(1-W_y),\qquad J_x\left(i_1+\tfrac{1}{2},j_1+1\right)=\frac{1}{\Delta x\Delta y}F_xW_y,$$
$$J_y\left(i_1,j_1+\tfrac{1}{2}\right)=\frac{1}{\Delta x\Delta y}F_y(1-W_x),\qquad J_y\left(i_1+1,j_1+\tfrac{1}{2}\right)=\frac{1}{\Delta x\Delta y}F_yW_x,$$

$i_1+1=i_2$ and $j_1+1=j_2$

$$i_1\equiv\text{floor}(x_1/\Delta x),\qquad i_2\equiv\text{floor}(x_2/\Delta x),$$
$$j_1\equiv\text{floor}(y_1/\Delta y),\qquad j_2\equiv\text{floor}(y_2/\Delta y),$$

$$F_x=q\frac{x_2-x_1}{\Delta t},\qquad F_y=q\frac{y_2-y_1}{\Delta t},$$

$$W_x=\frac{x_1+x_2}{2\Delta x}-i_1,\qquad W_y=\frac{y_1+y_2}{2\Delta y}-j_1.$$



see Umeda (2003) for detailed numerical method

Zigzag scheme in two-dimensional systems

$$\frac{J_x^{t+\Delta t/2}(i+\frac{1}{2},j)-J_x^{t+\Delta t/2}(i-\frac{1}{2},j)}{\Delta x}+\frac{J_y^{t+\frac{\Delta t}{2}}(i,j+\frac{1}{2})-J_y^{t+\Delta t/2}(i,j-\frac{1}{2})}{\Delta y}$$
$$=\frac{\rho^t(i,j)-\rho^{t+\Delta t}(i,j)}{\Delta t}.$$

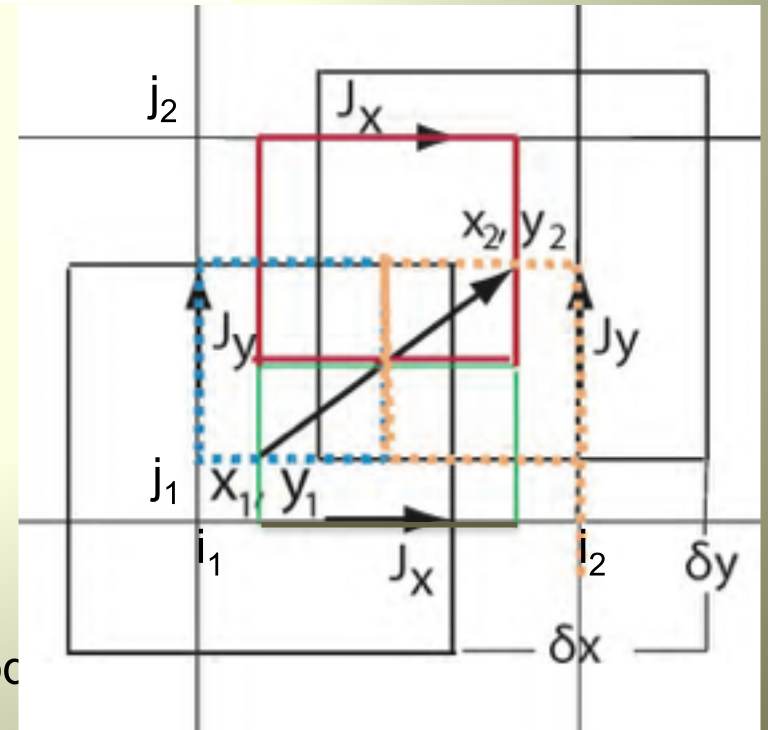
$$J_x\left(i_1+\tfrac{1}{2},j_1\right)=\frac{1}{\Delta x\Delta y}F_x(1-W_y),\qquad J_x\left(i_1+\tfrac{1}{2},j_1+1\right)=\frac{1}{\Delta x\Delta y}F_xW_y,$$
$$J_y\left(i_1,j_1+\tfrac{1}{2}\right)=\frac{1}{\Delta x\Delta y}F_y(1-W_x),\qquad J_y\left(i_1+1,j_1+\tfrac{1}{2}\right)=\frac{1}{\Delta x\Delta y}F_yW_x,$$

$i_1+1=i_2$ and $j_1+1=j_2$

$$i_1\equiv\mathrm{floor}(x_1/\Delta x),\qquad i_2\equiv\mathrm{floor}(x_2/\Delta x),$$
$$j_1\equiv\mathrm{floor}(y_1/\Delta y),\qquad j_2\equiv\mathrm{floor}(y_2/\Delta y),$$

$$F_x=q\frac{x_2-x_1}{\Delta t},\qquad F_y=q\frac{y_2-y_1}{\Delta t},$$

$$W_x=\frac{x_1+x_2}{2\Delta x}-i_1,\qquad W_y=\frac{y_1+y_2}{2\Delta y}-j_1.$$



see Umeda (2003) for detailed numerical method

Zigzag scheme in two-dimensional systems

$$\frac{J_x^{t+\Delta t/2}(i+\frac{1}{2},j) - J_x^{t+\Delta t/2}(i-\frac{1}{2},j)}{\Delta x} + \frac{J_y^{t+\frac{\Delta t}{2}}(i,j+\frac{1}{2}) - J_y^{t+\Delta t/2}(i,j-\frac{1}{2})}{\Delta y} = \frac{\rho^t(i,j) - \rho^{t+\Delta t}(i,j)}{\Delta t}.$$

$$J_x\left(i_1+\tfrac{1}{2}, j_1\right) = \frac{1}{\Delta x \Delta y} F_x (1 - W_y), \qquad J_x\left(i_1+\tfrac{1}{2}, j_1+1\right) = \frac{1}{\Delta x \Delta y} F_x W_y,$$

$$J_y\left(i_1, j_1+\tfrac{1}{2}\right) = \frac{1}{\Delta x \Delta y} F_y (1 - W_x), \qquad J_y\left(i_1+1, j_1+\tfrac{1}{2}\right) = \frac{1}{\Delta x \Delta y} F_y W_x,$$

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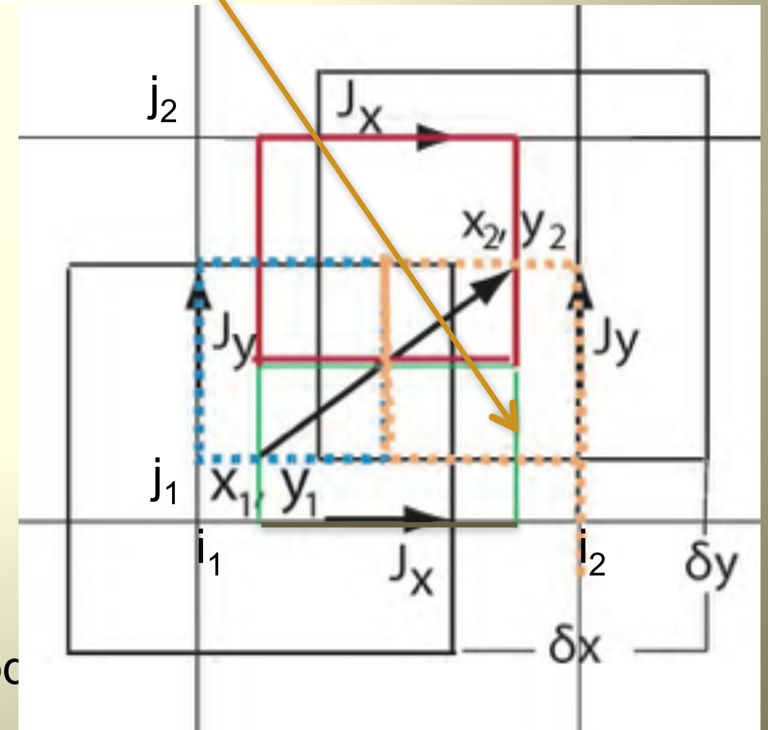

see Umeda (2003) for detailed numerical method

Zigzag scheme in two-dimensional systems

$$\frac{J_x^{t+\Delta t/2}(i+\frac{1}{2},j)-J_x^{t+\Delta t/2}(i-\frac{1}{2},j)}{\Delta x}+\frac{J_y^{t+\frac{\Delta t}{2}}(i,j+\frac{1}{2})-J_y^{t+\Delta t/2}(i,j-\frac{1}{2})}{\Delta y}$$
$$=\frac{\rho^t(i,j)-\rho^{t+\Delta t}(i,j)}{\Delta t}.$$

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$$J_y\left(i_1,j_1+\tfrac{1}{2}\right)=\frac{1}{\Delta x\Delta y}F_y(1-W_x),\qquad J_y\left(i_1+1,j_1+\tfrac{1}{2}\right)=\frac{1}{\Delta x\Delta y}F_yW_x,$$

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$$j_1\equiv\mathrm{floor}(y_1/\Delta y),\qquad j_2\equiv\mathrm{floor}(y_2/\Delta y),$$

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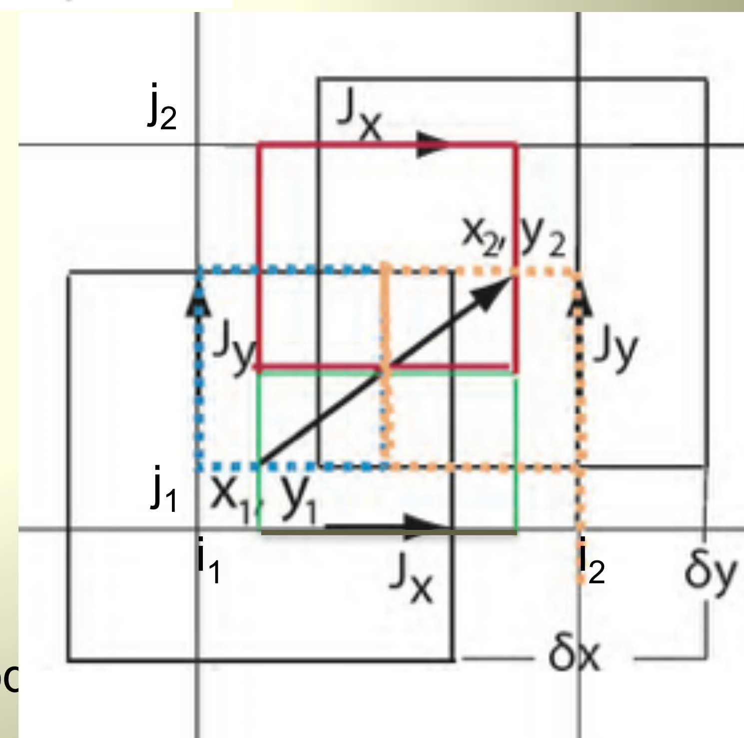
$$W_x=\frac{x_1+x_2}{2\Delta x}-i_1,\qquad W_y=\frac{y_1+y_2}{2\Delta y}-j_1.$$



see Umeda (2003) for detailed numerical method

Zigzag scheme in two-dimensional systems

$$\frac{J_x^{t+\Delta t/2}(i+\frac{1}{2},j)-J_x^{t+\Delta t/2}(i-\frac{1}{2},j)}{\Delta x}+\frac{J_y^{t+\frac{\Delta t}{2}}(i,j+\frac{1}{2})-J_y^{t+\Delta t/2}(i,j-\frac{1}{2})}{\Delta y} = \frac{\rho^t(i,j)-\rho^{t+\Delta t}(i,j)}{\Delta t}.$$

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$$J_y\left(i_1,j_1+\tfrac{1}{2}\right)=\frac{1}{\Delta x\Delta y}F_y(1-W_x), \qquad J_y\left(i_1+1,j_1+\tfrac{1}{2}\right)=\frac{1}{\Delta x\Delta y}F_yW_x,$$

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$$j_1 \equiv \mathrm{floor}(y_1/\Delta y), \qquad j_2 \equiv \mathrm{floor}(y_2/\Delta y),$$

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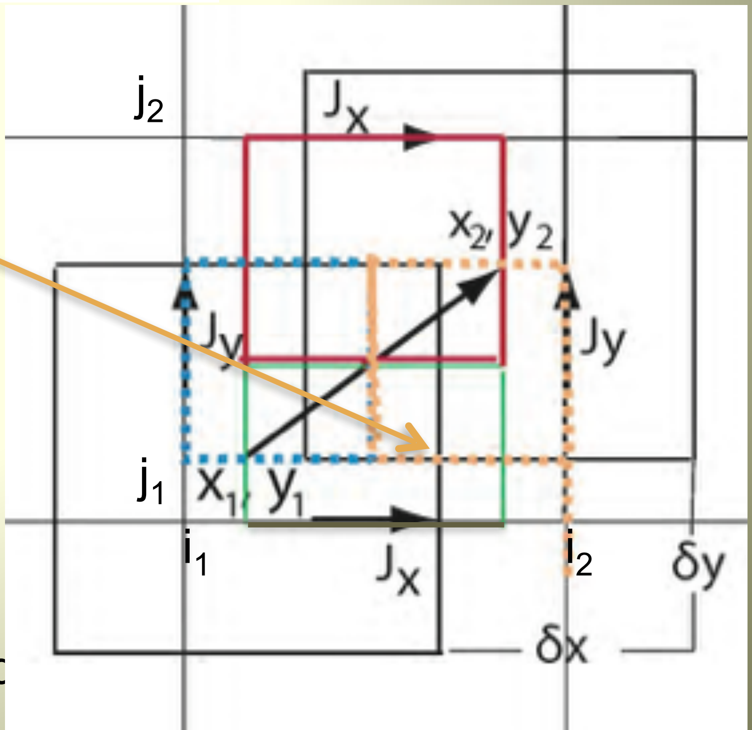
$$W_x = \frac{x_1+x_2}{2\Delta x}-i_1, \qquad W_y = \frac{y_1+y_2}{2\Delta y}-j_1.$$



see Umeda (2003) for detailed numerical method

Zigzag scheme in two-dimensional systems

$$\frac{J_x^{t+\Delta t/2}(i+\frac{1}{2},j)-J_x^{t+\Delta t/2}(i-\frac{1}{2},j)}{\Delta x}+\frac{J_y^{t+\frac{\Delta t}{2}}(i,j+\frac{1}{2})-J_y^{t+\Delta t/2}(i,j-\frac{1}{2})}{\Delta y}$$
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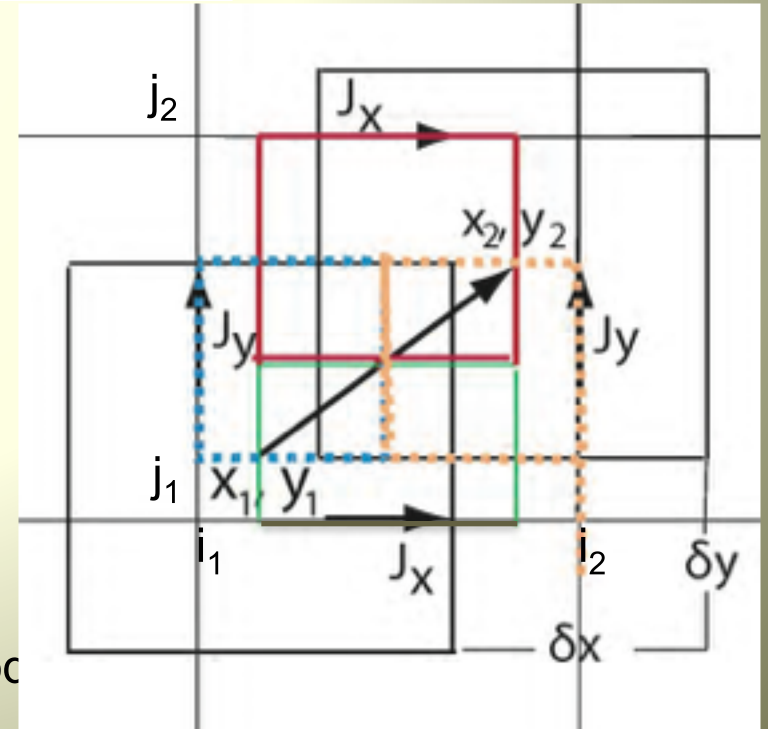

see Umeda (2003) for detailed numerical method

Zigzag scheme in two-dimensional systems

$$\frac{J_x^{t+\Delta t/2}(i+\frac{1}{2},j)-J_x^{t+\Delta t/2}(i-\frac{1}{2},j)}{\Delta x}+\frac{J_y^{t+\frac{\Delta t}{2}}(i,j+\frac{1}{2})-J_y^{t+\Delta t/2}(i,j-\frac{1}{2})}{\Delta y}$$
$$=\frac{\rho^t(i,j)-\rho^{t+\Delta t}(i,j)}{\Delta t}.$$

$$J_x\left(i_1+\tfrac{1}{2},j_1\right)=\frac{1}{\Delta x\Delta y}F_x(1-W_y),\qquad J_x\left(i_1+\tfrac{1}{2},j_1+1\right)=\frac{1}{\Delta x\Delta y}F_xW_y,$$
$$J_y\left(i_1,j_1+\tfrac{1}{2}\right)=\frac{1}{\Delta x\Delta y}F_y(1-W_x),\qquad J_y\left(i_1+1,j_1+\tfrac{1}{2}\right)=\frac{1}{\Delta x\Delta y}F_yW_x,$$

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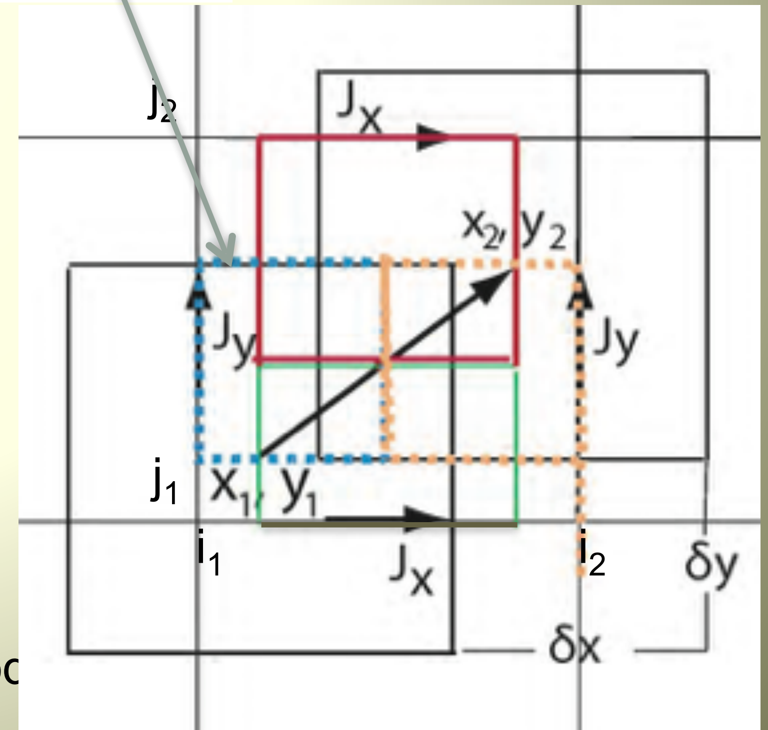
$$W_x=\frac{x_1+x_2}{2\Delta x}-i_1,\qquad W_y=\frac{y_1+y_2}{2\Delta y}-j_1.$$



see Umeda (2003) for detailed numerical method

Zigzag scheme in two-dimensional systems

$$\frac{J_x^{t+\Delta t/2}(i+\frac{1}{2},j) - J_x^{t+\Delta t/2}(i-\frac{1}{2},j)}{\Delta x} + \frac{J_y^{t+\frac{\Delta t}{2}}(i,j+\frac{1}{2}) - J_y^{t+\Delta t/2}(i,j-\frac{1}{2})}{\Delta y} = \frac{\rho^t(i,j) - \rho^{t+\Delta t}(i,j)}{\Delta t}.$$

$$J_x(i_1+\tfrac{1}{2}, j_1) = \frac{1}{\Delta x \Delta y} F_x(1-W_y), \qquad J_x(i_1+\tfrac{1}{2}, j_1+1) = \frac{1}{\Delta x \Delta y} F_x W_y,$$

$$J_y(i_1, j_1+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_y(1-W_x), \qquad J_y(i_1+1, j_1+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_y W_x,$$

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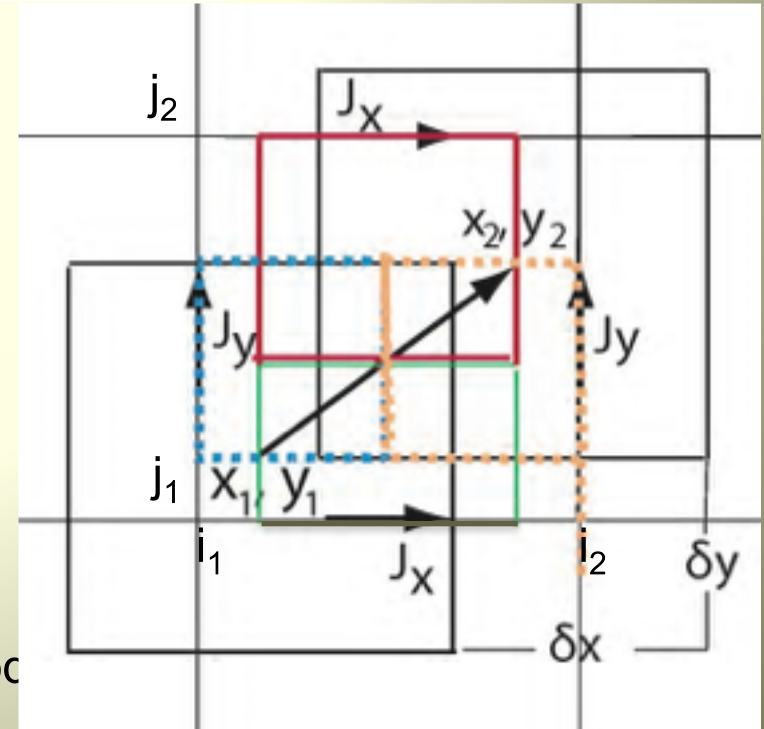
$$i_1 \equiv \mathrm{floor}(x_1/\Delta x), \qquad i_2 \equiv \mathrm{floor}(x_2/\Delta x),$$
$$j_1 \equiv \mathrm{floor}(y_1/\Delta y), \qquad j_2 \equiv \mathrm{floor}(y_2/\Delta y),$$

$$F_x = q\frac{x_2 - x_1}{\Delta t}, \qquad F_y = q\frac{y_2 - y_1}{\Delta t},$$

$$W_x = \frac{x_1 + x_2}{2\Delta x} - i_1, \qquad W_y = \frac{y_1 + y_2}{2\Delta y} - j_1.$$



see Umeda (2003) for detailed numerical method

New charge conservation method by Umeda

Charge flux of particle $q\boldsymbol{v} \equiv q(v_x, v_y, v_z)$

First-order shape-factor

$$S_i = \left\{ \begin{array}{ll} 1 - |\xi - i| & for\,|\xi - i| \le 1 \\ 0 & for\,|\xi - i| > 1 \end{array} \right\}$$

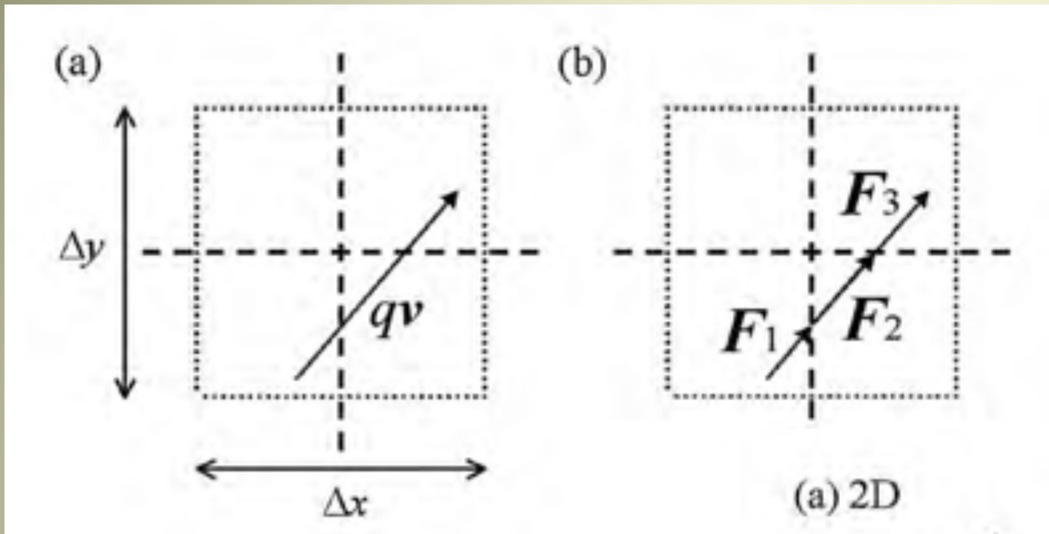
$$\rho(i,\ j,\ k) = \frac{1}{\Delta x \Delta y \Delta z} \sum_{n=1}^{N_p} q_n S_i(\frac{x_n}{\Delta x}) S_j(\frac{y_n}{\Delta y}) S_k(\frac{z_n}{\Delta z})$$

$$J_x(i + \tfrac{1}{2},\ j,\ k) = \frac{1}{\Delta x \Delta y \Delta z} \sum_{n=1}^{N_p} q_n v_{xn} S_{i+1/2}(\frac{x_n}{\Delta x}) S_j(\frac{y_n}{\Delta y}) S_k(\frac{z_n}{\Delta z})$$

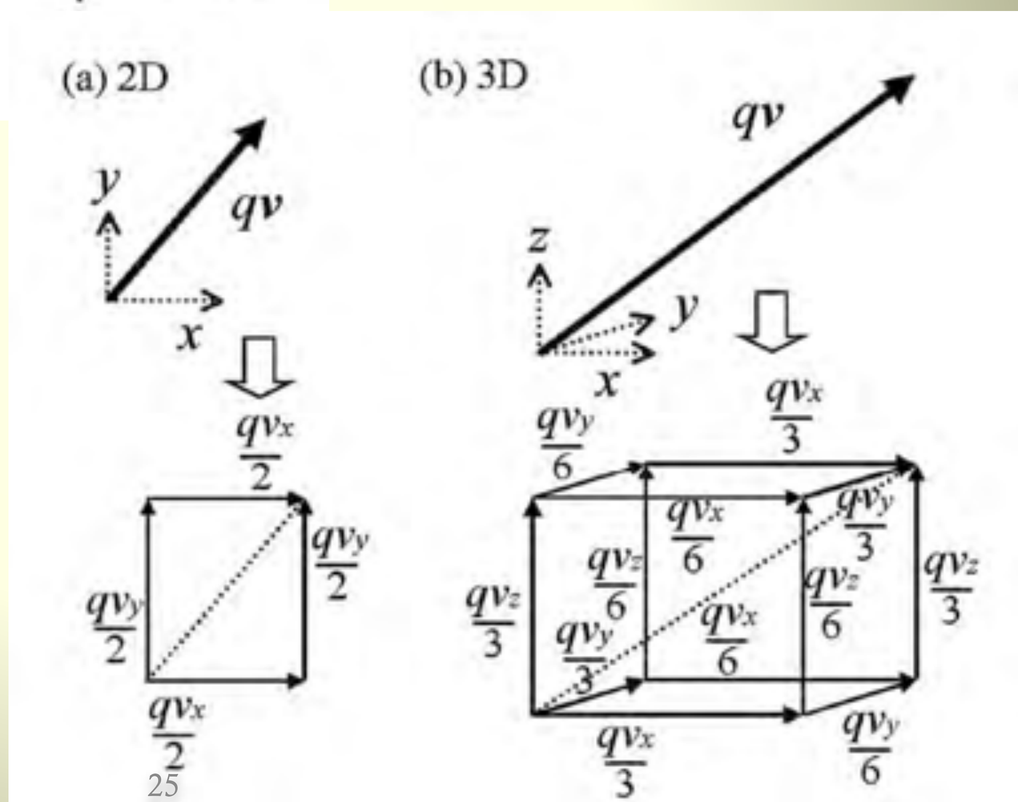
$$J_x(i,\ j + \tfrac{1}{2},\ k) = \frac{1}{\Delta x \Delta y \Delta z} \sum_{n=1}^{N_p} q_n v_{yn} S_i(\frac{x_n}{\Delta x}) S_{j+1/2}(\frac{y_n}{\Delta y}) S_k(\frac{z_n}{\Delta z})$$

$$J_z(i,\ j,\ k + \tfrac{1}{2}) = \frac{1}{\Delta x \Delta y \Delta z} \sum_{n=1}^{N_p} q_n v_{zn} S_i(\frac{x_n}{\Delta x}) S_j(\frac{y_n}{\Delta y}) S_{k+1/2}(\frac{z_n}{\Delta z})$$

Differences between two methods



$$q\boldsymbol{v} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3$$



Esirkepov's method is easier and implemented without IF statement

2D Zigzag method ($i_1 \neq i_2$ and $j_1 \neq j_2$)

$$x_r = \min\left[\min(i_1\Delta x, i_2\Delta x) + \Delta x, \max\left\{\max(i_1\Delta x, i_2\Delta x), \frac{x_1 + x_2}{2}\right\}\right]$$

$$y_r = \min\left[\min(j_1\Delta y, j_2\Delta y) + \Delta y, \max\left\{\max(j_1\Delta y, j_2\Delta y), \frac{y_1 + y_2}{2}\right\}\right]$$

$$\boldsymbol{F}_1 = (F_{x1}, F_{y1}) \quad \boldsymbol{F}_2 = (F_{x2}, F_{y2})$$

$$F_{x1} = q\frac{x_r - x_1}{\Delta t}, \; F_{y1} = q\frac{y_r - y_1}{\Delta t},$$

$$F_{x2} = q\frac{x_2 - x_r}{\Delta t} = qv_x - F_{x1}, \; F_{y2} = q\frac{y_2 - y_r}{\Delta t} = qv_y - F_{y1}$$

$$W_{x1} = \frac{x_1 + x_r}{2\Delta x} - i_1, \; W_{y1} = \frac{y_1 + y_r}{2\Delta y} - j_1,$$

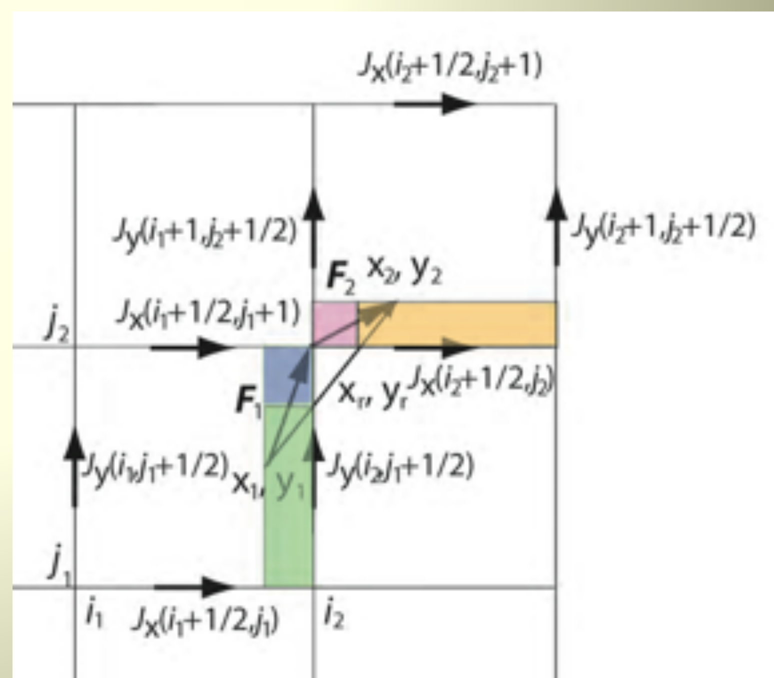
$$W_{x2} = \frac{x_r + x_2}{2\Delta x} - i_2, \; W_{y2} = \frac{y_r + y_2}{2\Delta y} - j_2,$$

$$J_x(i_1 + \tfrac{1}{2}, j_1) = \frac{1}{\Delta x \Delta y} F_{x1}(1 - W_{y1}), \; J_x(i_1 + \tfrac{1}{2}, j_1 + 1) = \frac{1}{\Delta x \Delta y} F_{x1} W_{y1},$$

$$J_y(i_1, j_1 + \tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y1}(1 - W_{x1}), \; J_y(i_1 + 1, j_1 + \tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y1} W_{x1},$$

$$J_x(i_2 + \tfrac{1}{2}, j_2) = \frac{1}{\Delta x \Delta y} F_{x2}(1 - W_{y2}), \; J_x(i_2 + \tfrac{1}{2}, j_2 + 1) = \frac{1}{\Delta x \Delta y} F_{x2} W_{y2},$$

$$J_y(i_2, j_2 + \tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y2}(1 - W_{x2}), \; J_y(i_2 + 1, j_2 + \tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y2} W_{x2},$$



$$i_1 + 1 = i_2$$

2D Zigzag method ($i_1 \neq i_2$ and $j_1 \neq j_2$)

$$x_r = \min\left[\min(i_1\Delta x, i_2\Delta x) + \Delta x, \max\left\{\max(i_1\Delta x, i_2\Delta x), \frac{x_1 + x_2}{2}\right\}\right]$$

$$y_r = \min\left[\min(j_1\Delta y, j_2\Delta y) + \Delta y, \max\left\{\max(j_1\Delta y, j_2\Delta y), \frac{y_1 + y_2}{2}\right\}\right]$$

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$$F_{x1} = q\frac{x_r - x_1}{\Delta t}, \quad F_{y1} = q\frac{y_r - y_1}{\Delta t},$$

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$$W_{x1} = \frac{x_1 + x_r}{2\Delta x} - i_1, \quad W_{y1} = \frac{y_1 + y_r}{2\Delta y} - j_1,$$

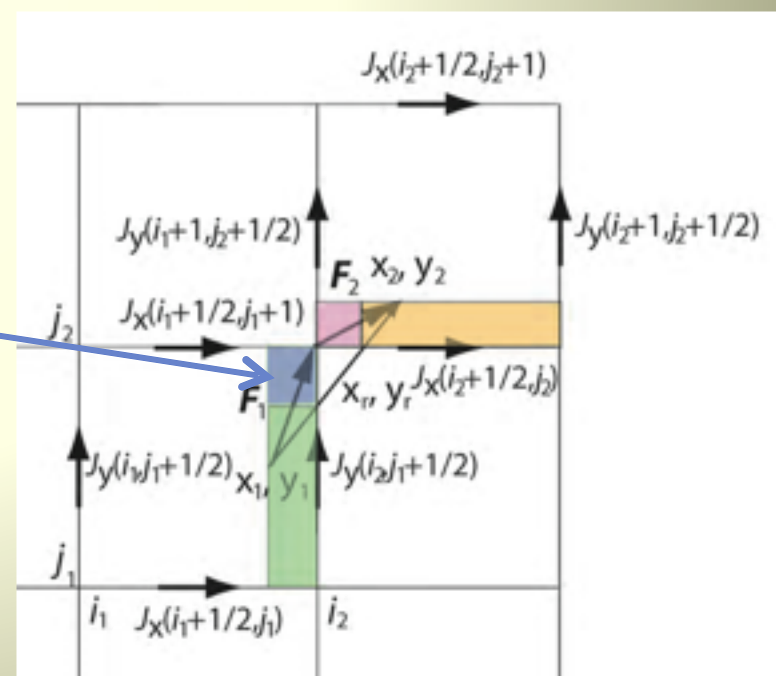
$$W_{x2} = \frac{x_r + x_2}{2\Delta x} - i_2, \quad W_{y2} = \frac{y_r + y_2}{2\Delta y} - j_2,$$

$$J_x(i_1 + \tfrac{1}{2}, j_1) = \frac{1}{\Delta x \Delta y} F_{x1}(1 - W_{y1}), \quad J_x(i_1 + \tfrac{1}{2}, j_1 + 1) = \frac{1}{\Delta x \Delta y} F_{x1} W_{y1},$$

$$J_y(i_1, j_1 + \tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y1}(1 - W_{x1}), \quad J_y(i_1 + 1, j_1 + \tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y1} W_{x1},$$

$$J_x(i_2 + \tfrac{1}{2}, j_2) = \frac{1}{\Delta x \Delta y} F_{x2}(1 - W_{y2}), \quad J_x(i_2 + \tfrac{1}{2}, j_2 + 1) = \frac{1}{\Delta x \Delta y} F_{x2} W_{y2},$$

$$J_y(i_2, j_2 + \tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y2}(1 - W_{x2}), \quad J_y(i_2 + 1, j_2 + \tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y2} W_{x2},$$



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$$y_r = \min\left[\min(j_1\Delta y, j_2\Delta y) + \Delta y, \max\left\{\max(j_1\Delta y, j_2\Delta y), \frac{y_1 + y_2}{2}\right\}\right]$$

$$\boldsymbol{F}_1 = (F_{x1}, F_{y1}) \quad \boldsymbol{F}_2 = (F_{x2}, F_{y2})$$

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$$J_x(i_2 + \tfrac{1}{2}, j_2) = \frac{1}{\Delta x \Delta y} F_{x2}(1 - W_{y2}), \quad J_x(i_2 + \tfrac{1}{2}, j_2 + 1) = \frac{1}{\Delta x \Delta y} F_{x2} W_{y2},$$

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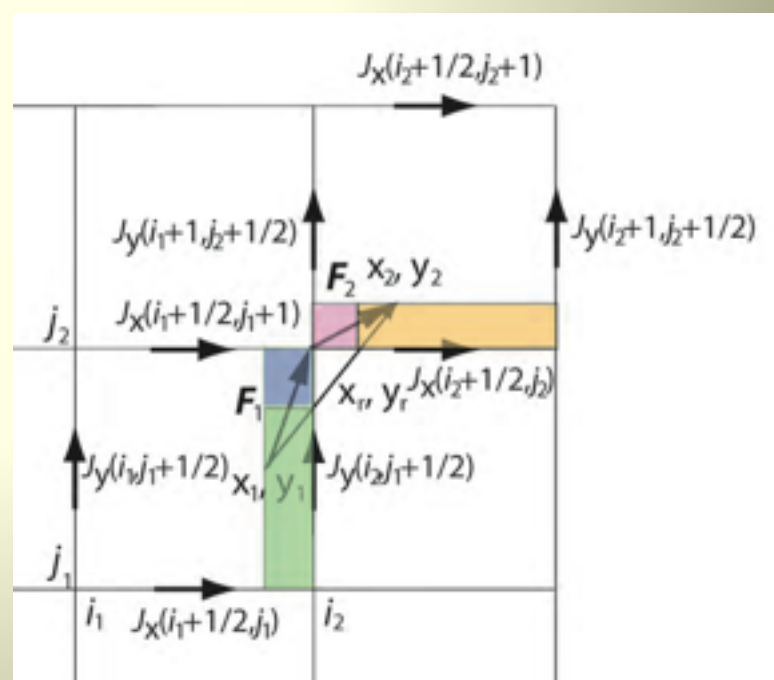
$$i_1 + 1 = i_2$$



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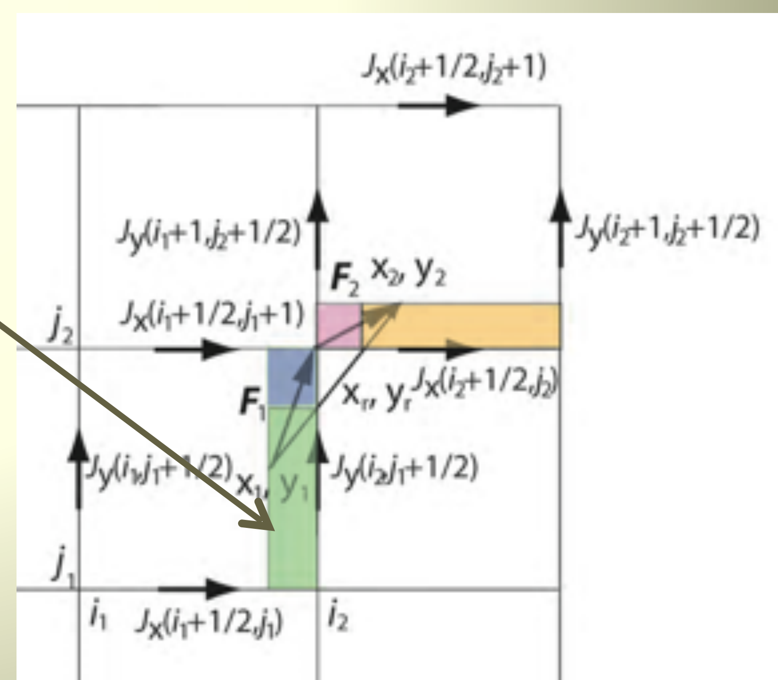

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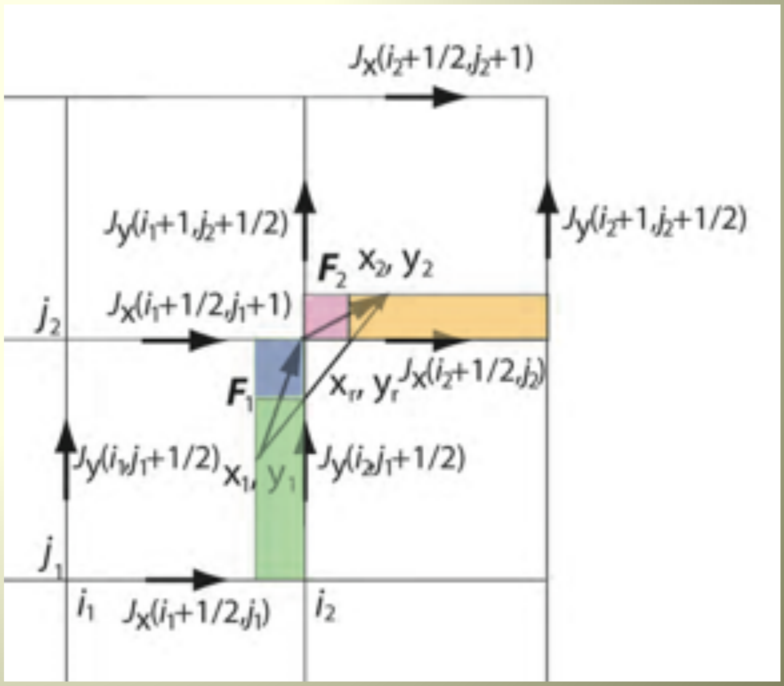

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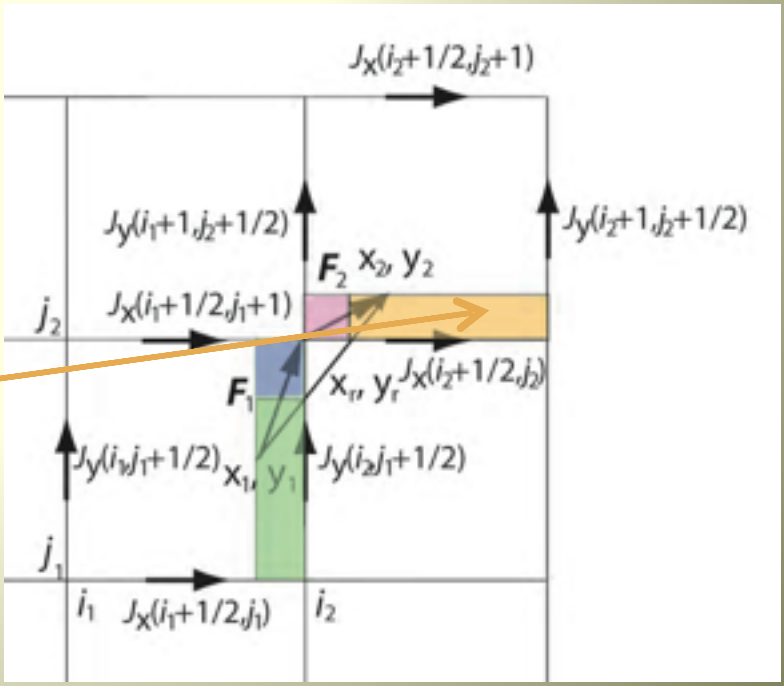

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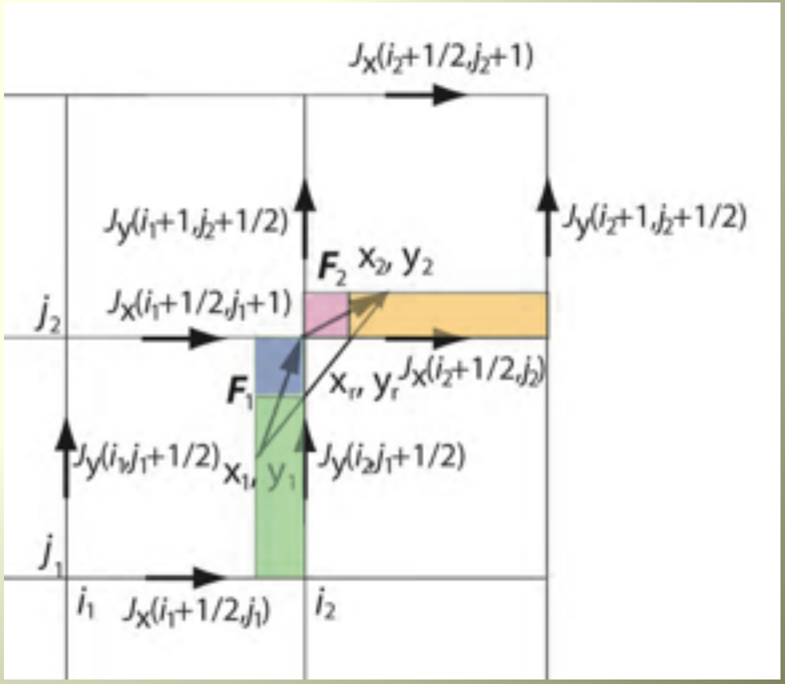

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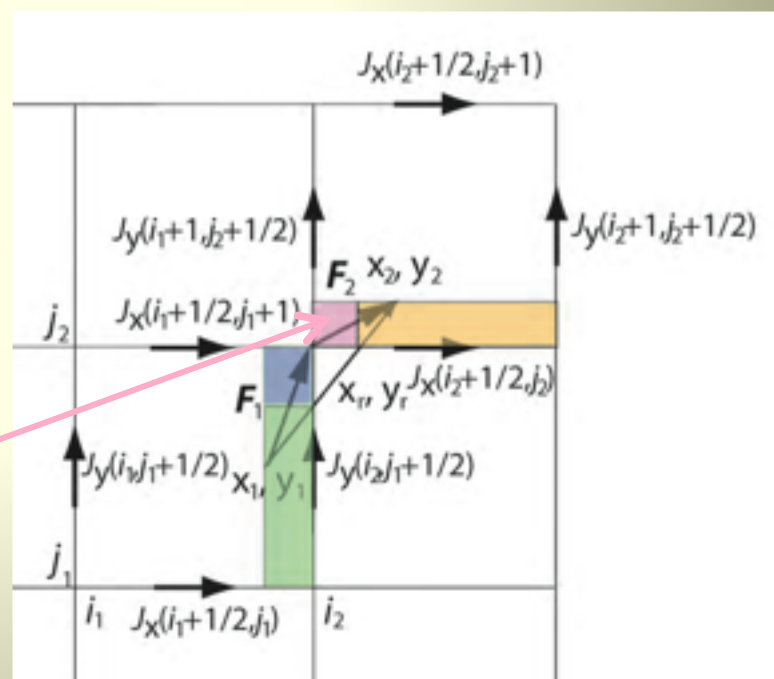

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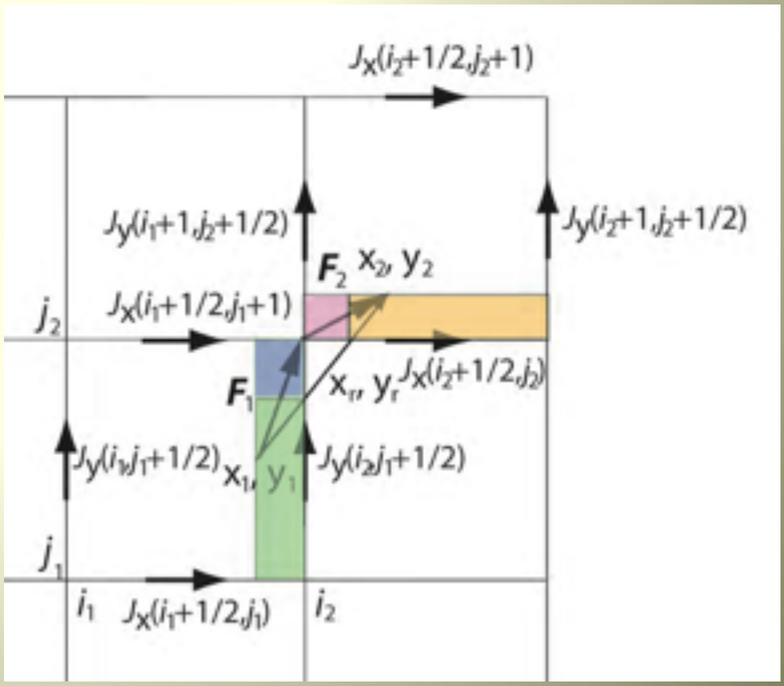

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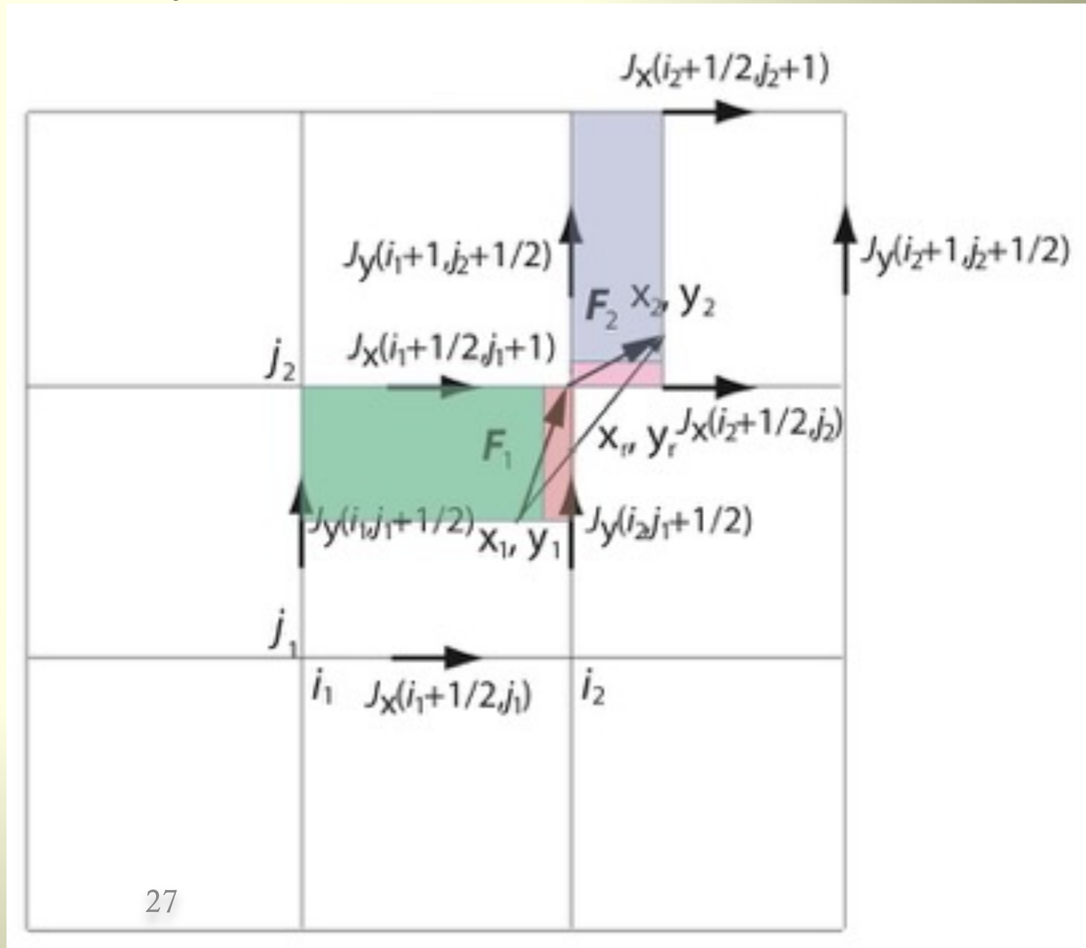

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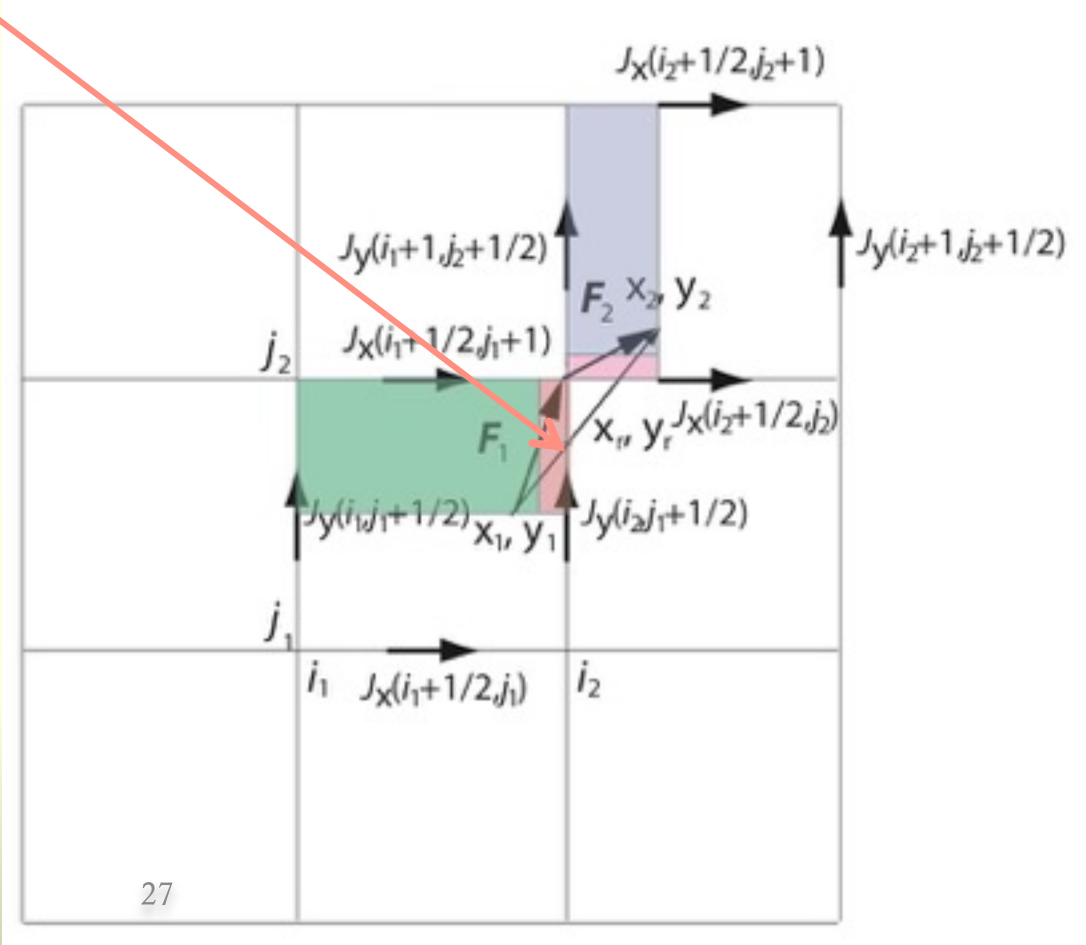

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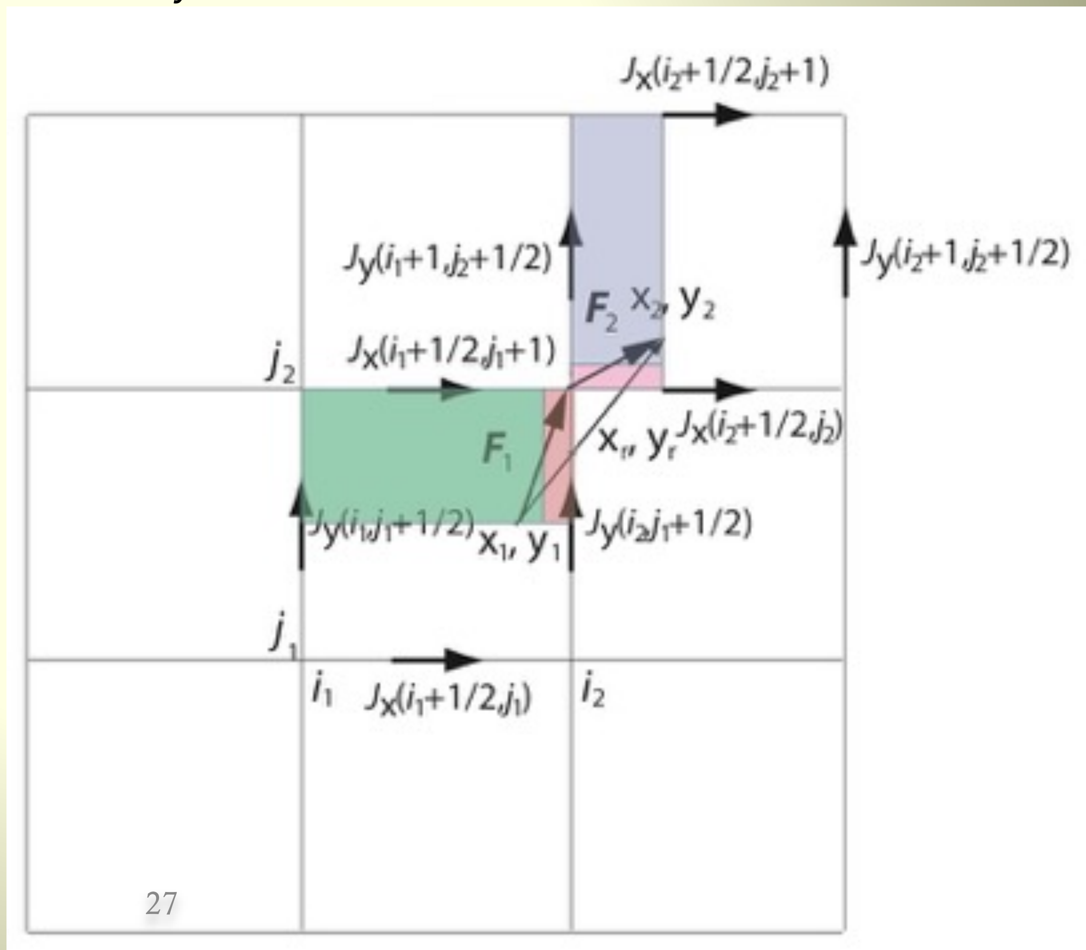
$$W_{x1} = \frac{x_1 + x_r}{2\Delta x} - i_1, \quad W_{y1} = \frac{y_1 + y_r}{2\Delta y} - j_1,$$

$$W_{x2} = \frac{x_r + x_2}{2\Delta x} - i_2, \quad W_{y2} = \frac{y_r + y_2}{2\Delta y} - j_2,$$



$$J_x(i_1+\tfrac{1}{2}, j_1) = \frac{1}{\Delta x \Delta y} F_{x1}(1-W_{y1}), \quad J_x(i_1+\tfrac{1}{2}, j_1+1) = \frac{1}{\Delta x \Delta y} F_{x1} W_{y1},$$

$$J_y(i_1, j_1+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y1}(1-W_{x1}), \quad J_y(i_1+1, j_1+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y1} W_{x1},$$

$$J_x(i_2+\tfrac{1}{2}, j_2) = \frac{1}{\Delta x \Delta y} F_{x2}(1-W_{y2}), \quad J_x(i_2+\tfrac{1}{2}, j_2+1) = \frac{1}{\Delta x \Delta y} F_{x2} W_{y2},$$

$$J_y(i_2, j_2+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y2}(1-W_{x2}), \quad J_y(i_2+1, j_2+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y2} W_{x2},$$

$$\boldsymbol{F}_1 = (F_{x1}, F_{y1}) \quad \boldsymbol{F}_2 = (F_{x2}, F_{y2})$$

$$F_{x1} = q\frac{x_r - x_1}{\Delta t}, \quad F_{y1} = q\frac{y_r - y_1}{\Delta t},$$

$$F_{x2} = q\frac{x_2 - x_r}{\Delta t}, \quad F_{y2} = q\frac{y_2 - y_r}{\Delta t}$$

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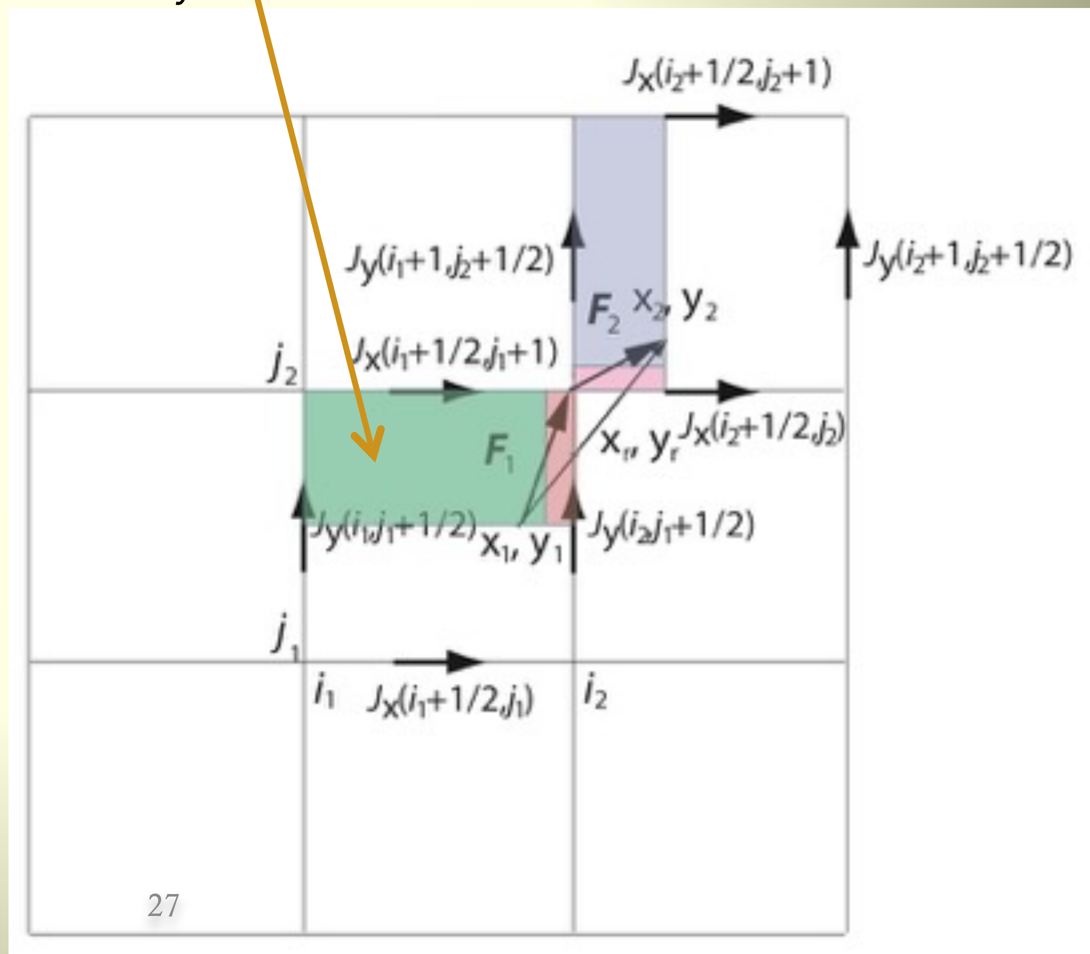

$$J_x(i_1+\tfrac{1}{2}, j_1) = \frac{1}{\Delta x \Delta y} F_{x1}(1-W_{y1}), \quad J_x(i_1+\tfrac{1}{2}, j_1+1) = \frac{1}{\Delta x \Delta y} F_{x1} W_{y1},$$

$$J_y(i_1, j_1+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y1}(1-W_{x1}), \quad J_y(i_1+1, j_1+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y1} W_{x1},$$

$$J_x(i_2+\tfrac{1}{2}, j_2) = \frac{1}{\Delta x \Delta y} F_{x2}(1-W_{y2}), \quad J_x(i_2+\tfrac{1}{2}, j_2+1) = \frac{1}{\Delta x \Delta y} F_{x2} W_{y2},$$

$$J_y(i_2, j_2+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y2}(1-W_{x2}), \quad J_y(i_2+1, j_2+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y2} W_{x2},$$

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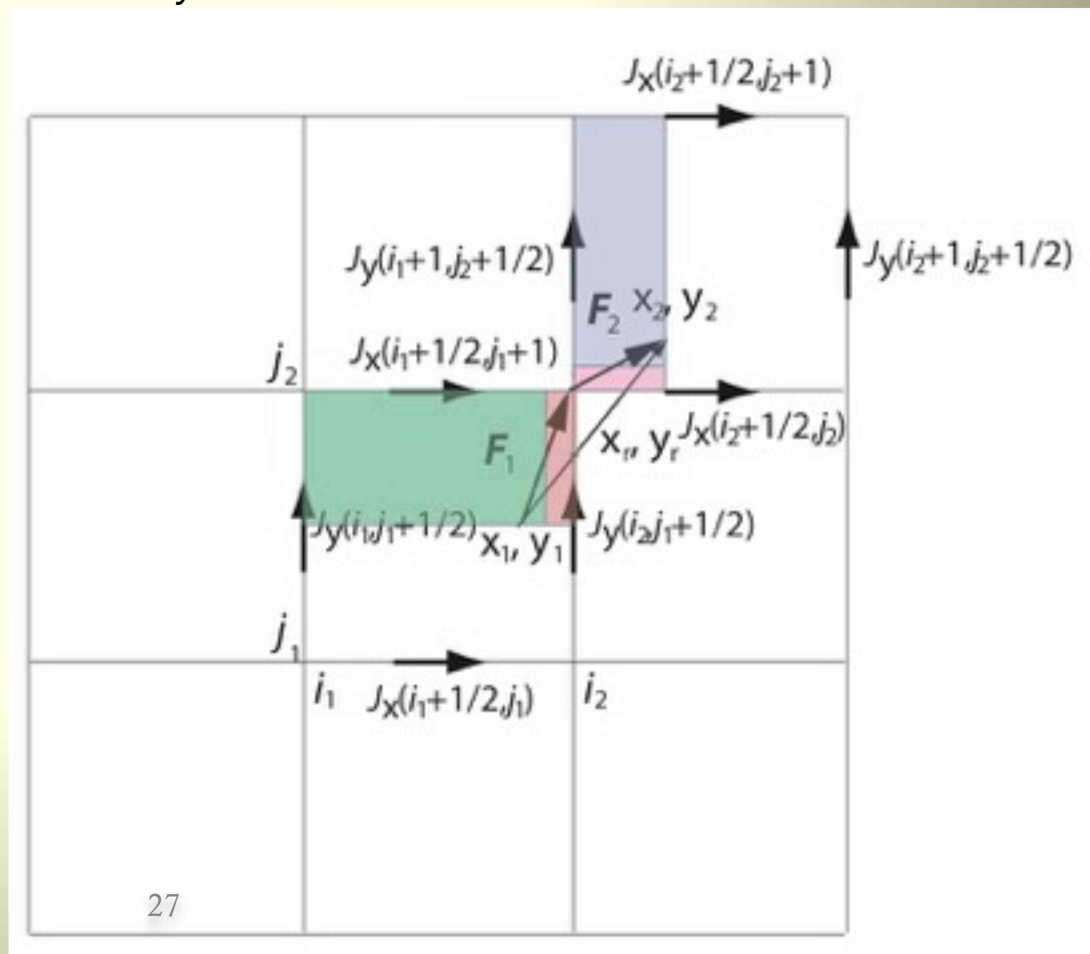

$$J_x(i_1+\tfrac{1}{2}, j_1) = \frac{1}{\Delta x \Delta y} F_{x1}(1-W_{y1}), \quad J_x(i_1+\tfrac{1}{2}, j_1+1) = \frac{1}{\Delta x \Delta y} F_{x1} W_{y1},$$

$$J_y(i_1, j_1+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y1}(1-W_{x1}), \quad J_y(i_1+1, j_1+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y1} W_{x1},$$

$$J_x(i_2+\tfrac{1}{2}, j_2) = \frac{1}{\Delta x \Delta y} F_{x2}(1-W_{y2}), \quad J_x(i_2+\tfrac{1}{2}, j_2+1) = \frac{1}{\Delta x \Delta y} F_{x2} W_{y2},$$

$$J_y(i_2, j_2+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y2}(1-W_{x2}), \quad J_y(i_2+1, j_2+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y2} W_{x2},$$

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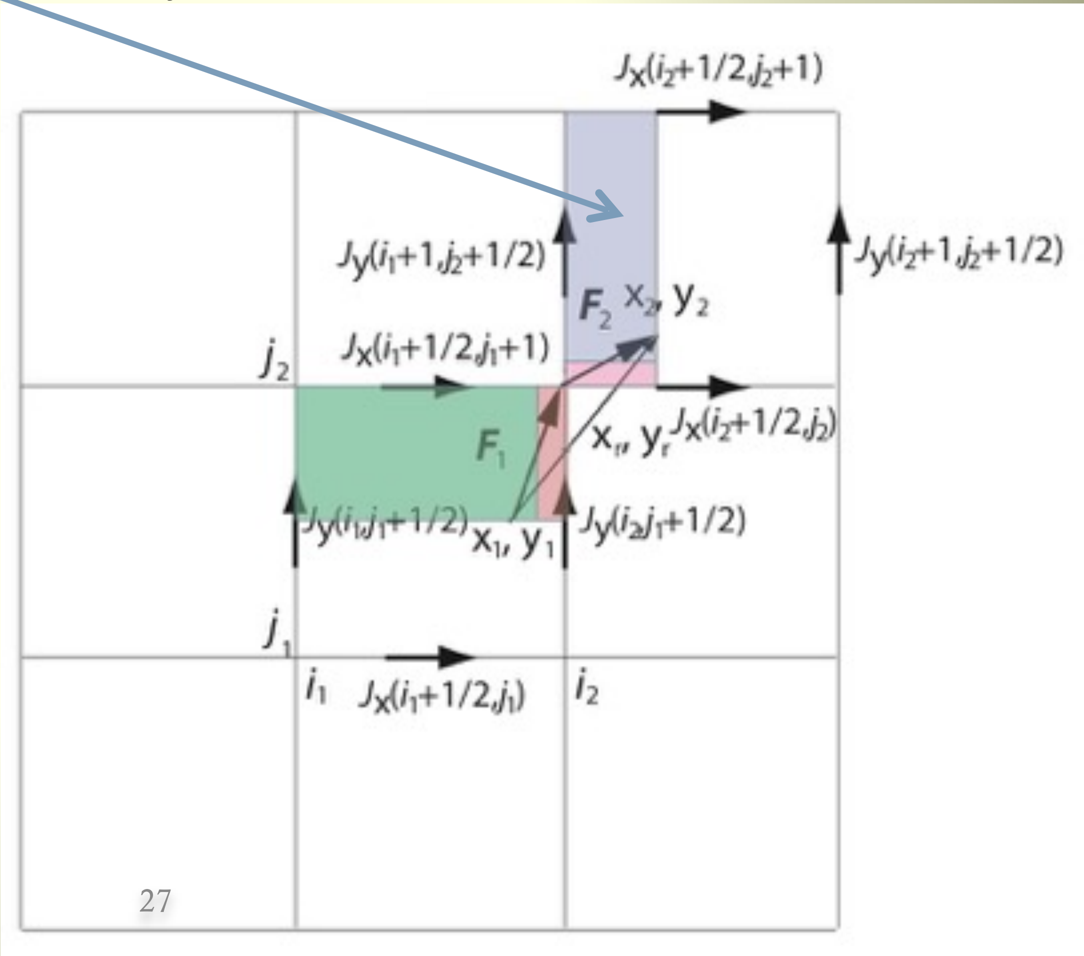

$$J_x(i_1+\tfrac{1}{2}, j_1) = \frac{1}{\Delta x \Delta y} F_{x1}(1-W_{y1}), \quad J_x(i_1+\tfrac{1}{2}, j_1+1) = \frac{1}{\Delta x \Delta y} F_{x1} W_{y1},$$

$$J_y(i_1, j_1+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y1}(1-W_{x1}), \quad J_y(i_1+1, j_1+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y1} W_{x1},$$

$$J_x(i_2+\tfrac{1}{2}, j_2) = \frac{1}{\Delta x \Delta y} F_{x2}(1-W_{y2}), \quad J_x(i_2+\tfrac{1}{2}, j_2+1) = \frac{1}{\Delta x \Delta y} F_{x2} W_{y2},$$

$$J_y(i_2, j_2+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y2}(1-W_{x2}), \quad J_y(i_2+1, j_2+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y2} W_{x2},$$

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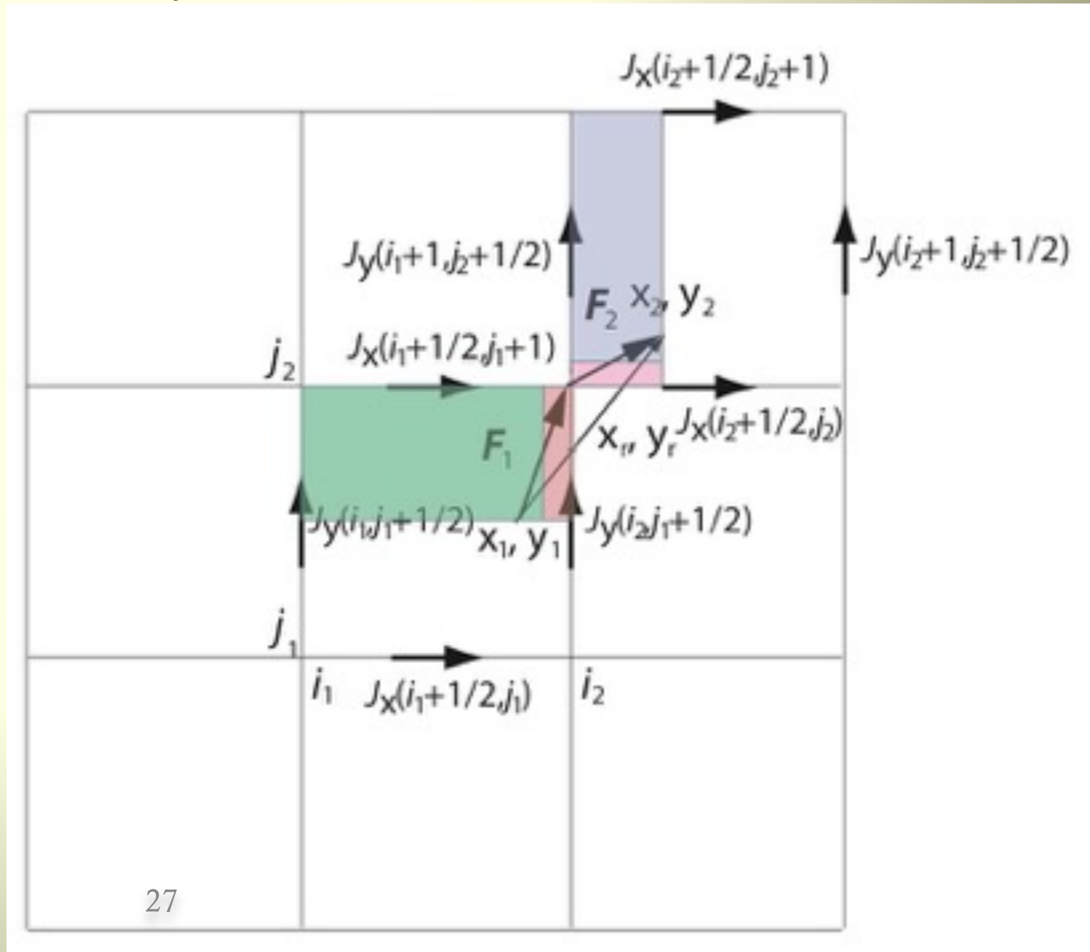

$$J_x(i_1+\tfrac{1}{2}, j_1) = \frac{1}{\Delta x \Delta y} F_{x1}(1-W_{y1}), \quad J_x(i_1+\tfrac{1}{2}, j_1+1) = \frac{1}{\Delta x \Delta y} F_{x1} W_{y1},$$

$$J_y(i_1, j_1+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y1}(1-W_{x1}), \quad J_y(i_1+1, j_1+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y1} W_{x1},$$

$$J_x(i_2+\tfrac{1}{2}, j_2) = \frac{1}{\Delta x \Delta y} F_{x2}(1-W_{y2}), \quad J_x(i_2+\tfrac{1}{2}, j_2+1) = \frac{1}{\Delta x \Delta y} F_{x2} W_{y2},$$

$$J_y(i_2, j_2+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y2}(1-W_{x2}), \quad J_y(i_2+1, j_2+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y2} W_{x2},$$

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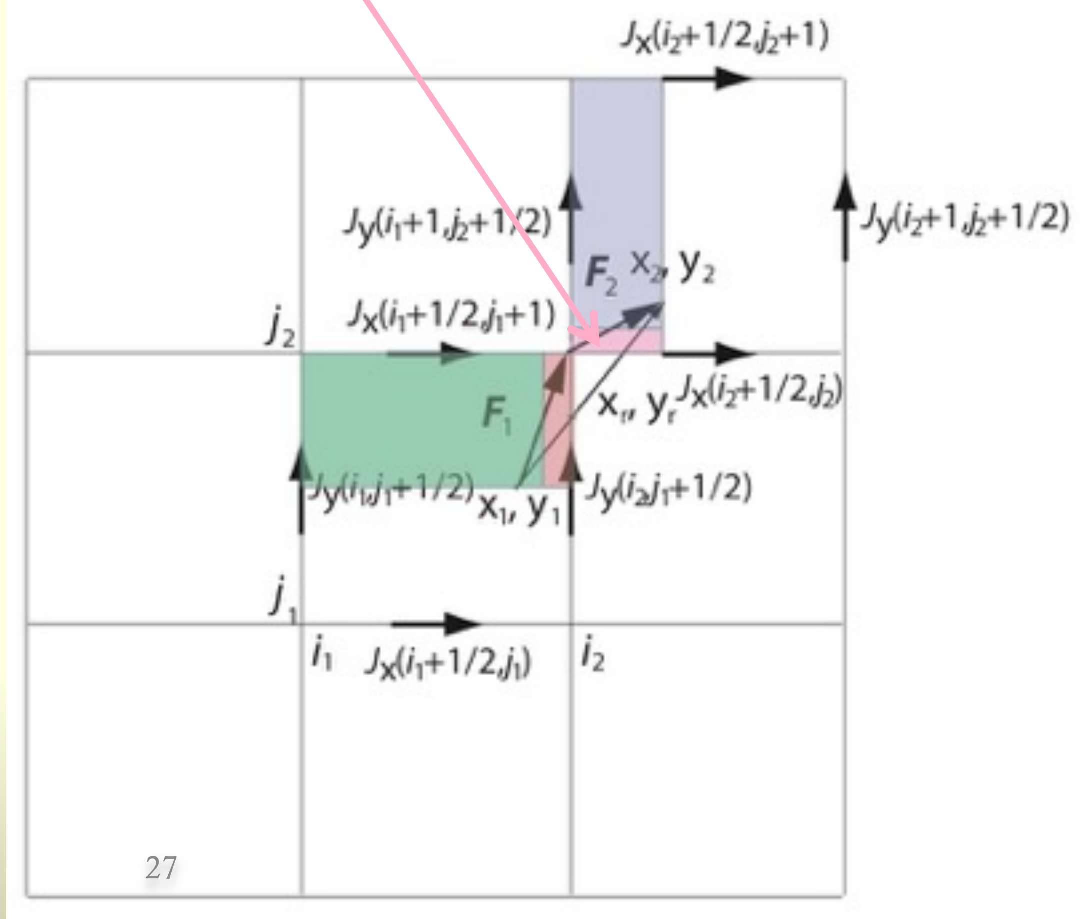

$$J_x(i_1+\tfrac{1}{2}, j_1) = \frac{1}{\Delta x \Delta y} F_{x1}(1-W_{y1}), \quad J_x(i_1+\tfrac{1}{2}, j_1+1) = \frac{1}{\Delta x \Delta y} F_{x1} W_{y1},$$

$$J_y(i_1, j_1+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y1}(1-W_{x1}), \quad J_y(i_1+1, j_1+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y1} W_{x1},$$

$$J_x(i_2+\tfrac{1}{2}, j_2) = \frac{1}{\Delta x \Delta y} F_{x2}(1-W_{y2}), \quad J_x(i_2+\tfrac{1}{2}, j_2+1) = \frac{1}{\Delta x \Delta y} F_{x2} W_{y2},$$

$$J_y(i_2, j_2+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y2}(1-W_{x2}), \quad J_y(i_2+1, j_2+\tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y2} W_{x2},$$

$$\boldsymbol{F}_1 = (F_{x1}, F_{y1}) \quad \boldsymbol{F}_2 = (F_{x2}, F_{y2})$$

$$F_{x1} = q\frac{x_r - x_1}{\Delta t}, \quad F_{y1} = q\frac{y_r - y_1}{\Delta t},$$

$$F_{x2} = q\frac{x_2 - x_r}{\Delta t}, \quad F_{y2} = q\frac{y_2 - y_r}{\Delta t}$$

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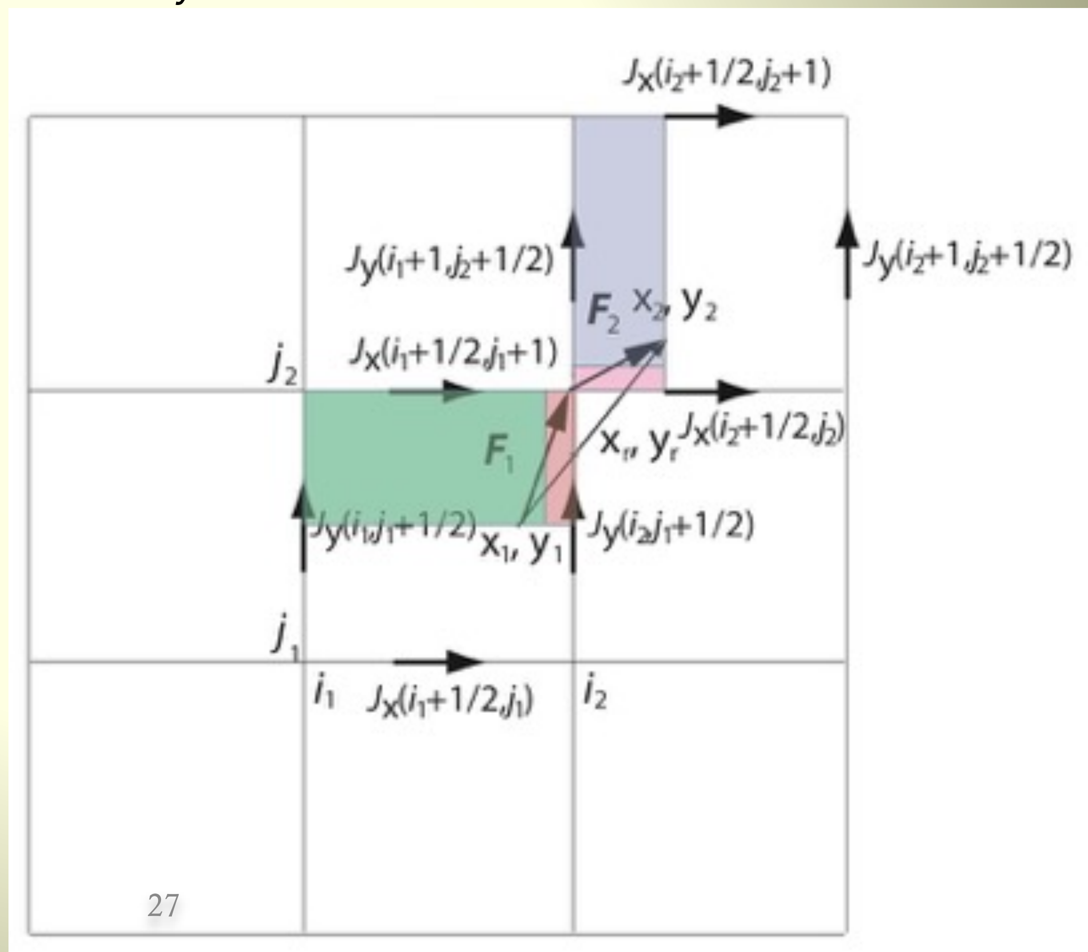


```
c ***************************************************************
      subroutine depsitUM1(x2,y2,z2,x1,y1,z1,dex,dey,dez,mFx,mFy,mFz,
     &                  q,DHDx,DHDy,DHDz)

      dimension dex(mFx,mFy,mFz),dey(mFx,mFy,mFz),dez(mFx,mFy,mFz)


      i1=x1 - DHDx          c  offset from the particle's nearest grid point
      j1=y1 - DHDy          c   in a VIRTUAL array
      k1=z1 - DHDz                DHDx = PBLeft-3.0
                                  DHDY = PBFrnt-3.0
      i2=x2 - DHDx                DHDz = PBBot -3.0
      j2=y2 - DHDy
      k2=z2 - DHDz

      xr = min(min(i1*1.0,i2*1.0)+1.0,max(max(i1*1.0,i2*1.0),
     &                  0.5*(x1+x2)-DHDx))
      yr = min(min(j1*1.0,j2*1.0)+1.0,max(max(j1*1.0,j2*1.0),
     &                  0.5*(y1+y2)-DHDy))
      zr = min(min(k1*1.0,k2*1.0)+1.0,max(max(k1*1.0,k2*1.0),
     &                  0.5*(z1+z2)-DHDz))
```

$$x_r = \min\left[\min(i_1\Delta x, i_2\Delta x) + \Delta x, \max\left\{\max(i_1\Delta x, i_2\Delta x), \frac{x_1 + x_2}{2}\right\}\right]$$

$$y_r = \min\left[\min(j_1\Delta y, j_2\Delta y) + \Delta y, \max\left\{\max(j_1\Delta y, j_2\Delta y), \frac{y_1 + y_2}{2}\right\}\right]$$

```
c      if (i1.eq.i2) then
c       xr = 0.5*(x1+x2)-DHDx
c      else
c       xr = max(i1,i2)
c      end if

c      if (j1.eq.j2) then
c       yr = 0.5*(y1+y2)-DHDy
c      else
c       yr = max(j1,j2)
c      end if

c      if (k1.eq.k2) then
c       zr = 0.5*(z1+z2)-DHDz
c      else
c       zr = max(k1,k2)
c      end if

       qu1=q*(xr-x1+DHDx)
       qv1=q*(yr-y1+DHDy)
       qw1=q*(zr-z1+DHDz)

       qu2=q*(x2-xr-DHDx)
       qv2=q*(y2-yr-DHDy)
       qw2=q*(z2-zr-DHDz)
```

$$\boldsymbol{F}_1 = (F_{x1}, F_{y1}) \quad \boldsymbol{F}_2 = (F_{x2}, F_{y2})$$

$$F_{x1} = q\frac{x_r - x_1}{\Delta t},\ F_{y1} = q\frac{y_r - y_1}{\Delta t},$$

$$F_{x2} = q\frac{x_2 - x_r}{\Delta t},\ F_{y2} = q\frac{y_2 - y_r}{\Delta t}$$

```
      dx1=0.5*(x1+xr-DHDx) - i1
      dy1=0.5*(y1+yr-DHDy) - j1
      dz1=0.5*(z1+zr-DHDz) - k1

      dx2=0.5*(x2+xr-DHDx) - i2
      dy2=0.5*(y2+yr-DHDy) - j2
      dz2=0.5*(z2+zr-DHDz) - k2
```

$$W_{x1} = \frac{x_1 + x_r}{2\Delta x} - i_1,\quad W_{y1} = \frac{y_1 + y_r}{2\Delta y} - j_1,$$

$$W_{x2} = \frac{x_r + x_2}{2\Delta x} - i_2,\quad W_{y2} = \frac{y_r + y_2}{2\Delta y} - j_2,$$

```
c  *** DEPOSIT CURRENT and SMOOTHING ***
c !!! NO SMOOTHING IS APPLIED HERE !!!
c attention
       if(i1.le.0) then
      write(6,*) 'i1,x1,DHDx,y1,DHDy,z1,DHDz=',
     1            i1,x1,DHDx,y1,DHDy,z1,DHDz,
     1           'i2,x2,DHDx,y2,DHDy,z2,DHDz=',
     2            i2,x2,DHDx,y2,DHDy,z2,DHDz
       endif
```

$$J_x(i_1 + \tfrac{1}{2}, j_1) = \frac{1}{\Delta x \Delta y} F_{x1}(1 - W_{y1}),\quad J_x(i_1 + \tfrac{1}{2}, j_1 + 1) = \frac{1}{\Delta x \Delta y} F_{x1} W_{y1},$$

$$J_y(i_1, j_1 + \tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y1}(1 - W_{x1}),\quad J_y(i_1 + 1, j_1 + \tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y1} W_{x1},$$

$$\frac{\partial \mathbf{E}}{\partial t} = c\nabla \times \mathbf{B} - \mathbf{J}$$

$$J_x(i_2 + \tfrac{1}{2}, j_2) = \frac{1}{\Delta x \Delta y} F_{x2}(1 - W_{y2}),\quad J_x(i_2 + \tfrac{1}{2}, j_2 + 1) = \frac{1}{\Delta x \Delta y} F_{x2} W_{y2},$$

$$J_y(i_2, j_2 + \tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y2}(1 - W_{x2}),\quad J_y(i_2 + 1, j_2 + \tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y2} W_{x2},$$

```
dx1=0.5*(x1+xr-DHDx) - i1
dy1=0.5*(y1+yr-DHDy) - j1
dz1=0.5*(z1+zr-DHDz) - k1

dx2=0.5*(x2+xr-DHDx) - i2
dy2=0.5*(y2+yr-DHDy) - j2
dz2=0.5*(z2+zr-DHDz) - k2
```

$$W_{x1} = \frac{x_1 + x_r}{2\Delta x} - i_1, \quad W_{y1} = \frac{y_1 + y_r}{2\Delta y} - j_1,$$

$$W_{x2} = \frac{x_r + x_2}{2\Delta x} - i_2, \quad W_{y2} = \frac{y_r + y_2}{2\Delta y} - j_2,$$

```
c  *** DEPOSIT CURRENT and SMOOTHING ***
c !!! NO SMOOTHING IS APPLIED HERE !!!
c attention
      if(i1.le.0) then
     write(6,*) 'i1,x1,DHDx,y1,DHDy,z1,DHDz=',
    1           i1,x1,DHDx,y1,DHDy,z1,DHDz,
    1          'i2,x2,DHDx,y2,DHDy,z2,DHDz=',
    2           i2,x2,DHDx,y2,DHDy,z2,DHDz
      endif
```

$$J_x(i_1 + \tfrac{1}{2}, j_1) = \frac{1}{\Delta x \Delta y} F_{x1}(1 - W_{y1}), \quad J_x(i_1 + \tfrac{1}{2}, j_1 + 1) = \frac{1}{\Delta x \Delta y} F_{x1} W_{y1},$$

$$J_y(i_1, j_1 + \tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y1}(1 - W_{x1}), \quad J_y(i_1 + 1, j_1 + \tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y1} W_{x1},$$

$$J_x(i_2 + \tfrac{1}{2}, j_2) = \frac{1}{\Delta x \Delta y} F_{x2}(1 - W_{y2}), \quad J_x(i_2 + \tfrac{1}{2}, j_2 + 1) = \frac{1}{\Delta x \Delta y} F_{x2} W_{y2},$$

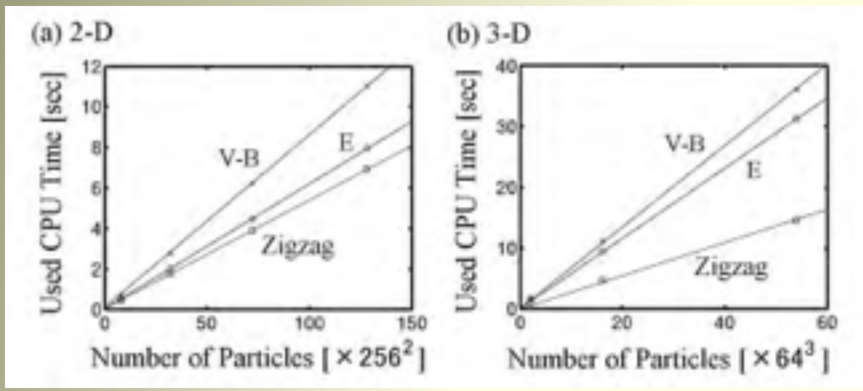
$$J_y(i_2, j_2 + \tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y2}(1 - W_{x2}), \quad J_y(i_2 + 1, j_2 + \tfrac{1}{2}) = \frac{1}{\Delta x \Delta y} F_{y2} W_{x2},$$

$$\frac{\partial \mathbf{E}}{\partial t} = \cancel{c\nabla \times \mathbf{B}} - \mathbf{J}$$

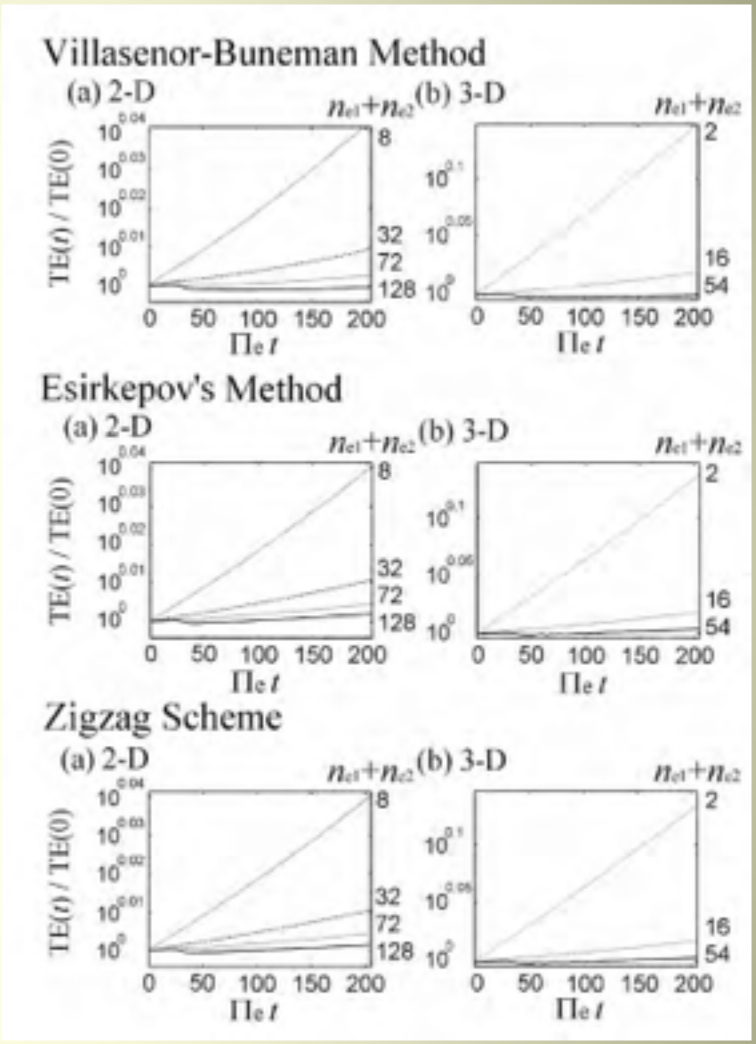
```
dex(i1,j1+1,k1+1)= dex(i1,j1+1,k1+1) - qu1*dy1*dz1
dex(i1,j1,k1+1)  = dex(i1,j1,k1+1)   - qu1*(1.-dy1)*dz1
dex(i1,j1+1,k1)  = dex(i1,j1+1,k1)   - qu1*dy1*(1.-dz1)
dex(i1,j1,k1)    = dex(i1,j1,k1)     - qu1*(1.-dy1)*(1.-dz1)

dey(i1+1,j1,k1+1)= dey(i1+1,j1,k1+1) - qv1*dx1*dz1
dey(i1+1,j1,k1)  = dey(i1+1,j1,k1)   - qv1*dx1*(1.-dz1)
dey(i1,j1,k1+1)  = dey(i1,j1,k1+1)   - qv1*(1.-dx1)*dz1
dey(i1,j1,k1)    = dey(i1,j1,k1)     - qv1*(1.-dx1)*(1.-dz1)

dez(i1+1,j1+1,k1)= dez(i1+1,j1+1,k1) - qw1*dx1*dy1
dez(i1,j1+1,k1)  = dez(i1,j1+1,k1)   - qw1*(1.-dx1)*dy1
dez(i1+1,j1,k1)  = dez(i1+1,j1,k1)   - qw1*dx1*(1.-dy1)
dez(i1,j1,k1)    = dez(i1,j1,k1)     - qw1*(1.-dx1)*(1.-dy1)


dex(i2,j2+1,k2+1)= dex(i2,j2+1,k2+1) - qu2*dy2*dz2
dex(i2,j2,k2+1)  = dex(i2,j2,k2+1)   - qu2*(1.-dy2)*dz2
dex(i2,j2+1,k2)  = dex(i2,j2+1,k2)   - qu2*dy2*(1.-dz2)
dex(i2,j2,k2)    = dex(i2,j2,k2)     - qu2*(1.-dy2)*(1.-dz2)

dey(i2+1,j2,k2+1)= dey(i2+1,j2,k2+1) - qv2*dx2*dz2
dey(i2+1,j2,k2)  = dey(i2+1,j2,k2)   - qv2*dx2*(1.-dz2)
dey(i2,j2,k2+1)  = dey(i2,j2,k2+1)   - qv2*(1.-dx2)*dz2
dey(i2,j2,k2)    = dey(i2,j2,k2)     - qv2*(1.-dx2)*(1.-dz2)
```

```
dez(i2+1,j2+1,k2)= dez(i2+1,j2+1,k2) - qw2*dx2*dy2
dez(i2,j2+1,k2)  = dez(i2,j2+1,k2)   - qw2*(1.-dx2)*dy2
dez(i2+1,j2,k2)  = dez(i2+1,j2,k2)   - qw2*dx2*(1.-dy2)
dez(i2,j2,k2)    = dez(i2,j2,k2)     - qw2*(1.-dx2)*(1.-dy2)

return
end
```

Comparisons among three different methods



Average CPU time used for the computation of current density for one time step



Total energy history for three methods with different numbers of particle per cell