Computational Methods for Kinetic Processes in Plasma Physics

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Context

- Cold plasma oscillation analysis
- Warm plasma of finite-size particles
- Loading a Maxwellian velocity distribution
- Loading drifting relativistic Maxwellian distribution by Singe

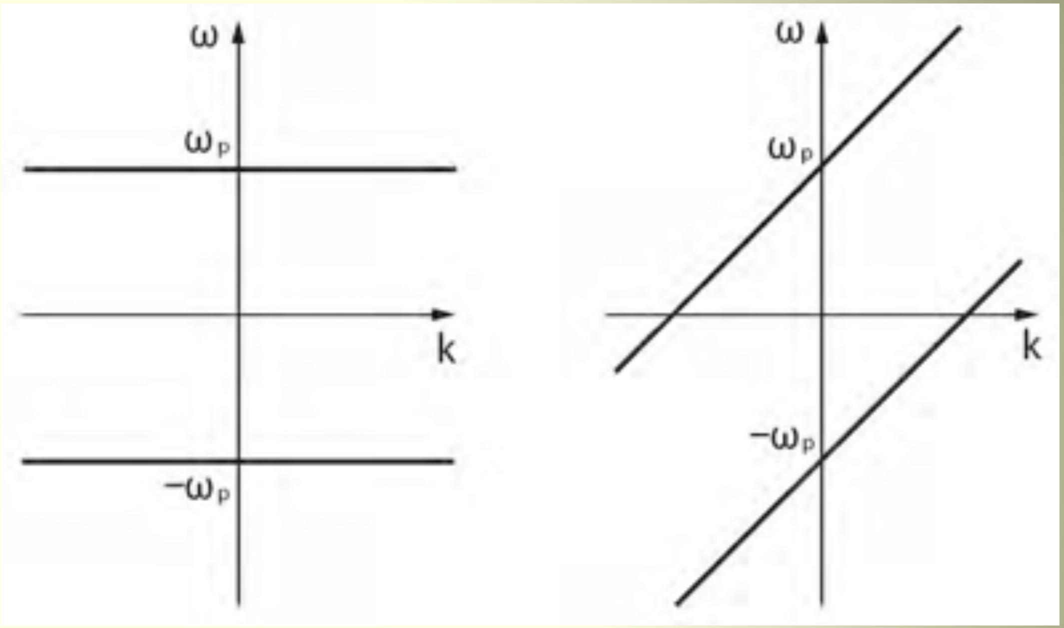
Cold plasma oscillation analysis with drift velocity (BL p.89)

$$m\frac{dv}{dt} = aE = F$$

$$\nabla \cdot \rho v + \frac{\partial \rho}{\partial t} = 0, \quad J = \rho v$$

$$\nabla \times H = J + \varepsilon_0 \frac{\partial E}{\partial t}$$

$$v(x,t) = v_0 + v_1(x,t), \quad \frac{d}{dt} = \frac{\partial}{\partial t} + v \cdot \nabla$$



after some calculation

$$\frac{\varepsilon}{\varepsilon_0} = 1 - \frac{\omega_{pe}^2}{(\omega - \boldsymbol{k} \cdot v_0)^2} = 0$$

$$\omega = \pm\omega_{pe} \qquad \omega = kv_0 \pm \omega_{pe}$$

Warm plasma of finite-size particles (BL p.68)

$\boldsymbol{E}_0 = \boldsymbol{B}_0 = 0$ and $f(\boldsymbol{r},\boldsymbol{v},t) = f_0(\boldsymbol{r},\boldsymbol{v},t) + f_1(\boldsymbol{r},\boldsymbol{v},t)$

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla f_1 - \frac{e}{m}\boldsymbol{E}_1\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} = 0$$

$$(\mathrm{Re}\,\omega)^2 \approx S^2(k)\omega_p^2 + 3k^2 v_t^2$$

$$\mathrm{Im}\,\omega \approx -\left(\frac{\pi}{8}\right)^{1/2} S\omega_p \left(\frac{S}{k\lambda_D}\right)^3 \exp\left[-\frac{1}{2}\left(\frac{S}{k\lambda_D}\right)^2 - \frac{3}{2}\right]$$

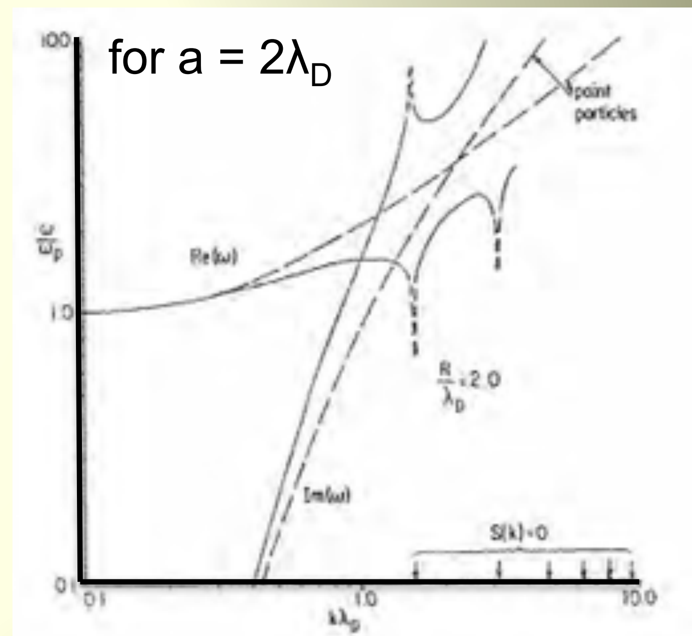
$$\frac{\partial f_1}{\partial t} + \boldsymbol{v}\cdot\nabla f_1 - \frac{e}{m}\boldsymbol{E}_1\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} = 0$$

$$f_1 \propto e^{i(kx-\omega t)}$$

$$\varepsilon_0 \nabla\cdot E_1 = ik\varepsilon_0 E_1 = -en_1 = -e\int f_1\, d\boldsymbol{v}$$

$$\varepsilon(k,\omega) = 1 - S^2(k)\frac{\omega_p^2}{k^2}\int \boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}\frac{d\boldsymbol{v}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}$$

$$\varepsilon(k,\omega) = 1 - \frac{1}{2}\left(\frac{S\omega_p}{kv_t}\right)^2 Z'\left(\frac{\omega}{\sqrt{2}kv_t}\right),\ \zeta = \frac{\omega}{\sqrt{2}kv_t}$$



$$Z(\zeta) = \frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}\frac{e^{-s^2}}{s-\zeta}ds, \quad \mathrm{Im}(\zeta) > 0$$

$$Z'(\zeta) = \frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}\frac{e^{-s^2}}{(s-\zeta)^2}ds = \frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}\frac{(d/ds)(e^{-s^2})}{s-\zeta}ds = -\frac{2}{\pi^{1/2}}\int_{-\infty}^{\infty}\frac{s(e^{-s^2})}{s-\zeta}ds$$

$$\varepsilon(k,\omega) = 1 - S^2(k)\frac{\omega_p^2}{k^2}\int \boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}\frac{d\boldsymbol{v}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}$$

$$f_0 = e^{-v^2/2v_t^2}$$

$$\frac{\partial f_0}{\partial \boldsymbol{v}} = -\frac{v}{v_t^2} f_0 \qquad v' = v/\sqrt{2}v_t$$

$$\int \boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}\frac{d\boldsymbol{v}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}} = -\frac{1}{v_t^2}\int f_0 \frac{2v_t^2 k v' d\boldsymbol{v}'}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}'\sqrt{2}v_t} = -\frac{1}{v_t^2}\int f_0 \frac{\sqrt{2}v_t v' d\boldsymbol{v}'}{\omega/\sqrt{2}kv_t - \boldsymbol{v}'}$$

$$= \frac{1}{v_t^2}\int f_0 \frac{\sqrt{2}v_t v' d\boldsymbol{v}'}{\boldsymbol{v}' - \omega/\sqrt{2}kv_t}$$

Two-beam instability: Linear analysis (BL p. 94)

Counter-streaming electron beams $v_{01} = -v_{01}$

$$\varepsilon(\omega, k) = 1 - \frac{\omega_{p1}^2}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v}_{01})^2} - \frac{\omega_{p2}^2}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v}_{02})^2}$$

$$\text{if } v_{01} = -v_{02} = v_0 \text{ and } \omega_{p1} = \omega_{p2} = \omega_p$$

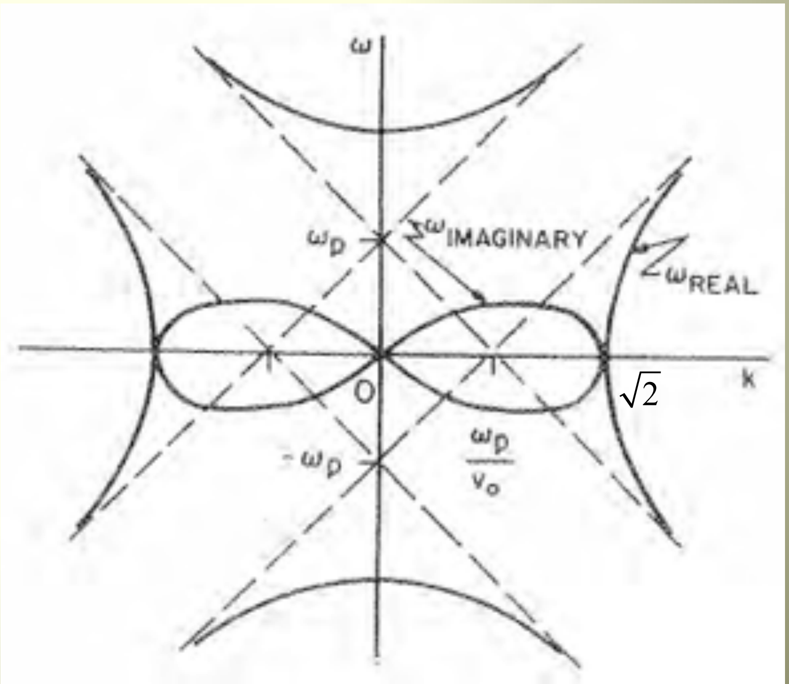
$$\omega = \pm[k^2 v_0^2 + \omega_p^2 \pm \omega_p (4k^2 v_0^2 + \omega_p^2)^{1/2}]^{1/2}$$

$\sqrt{2} < \frac{kv_0}{\omega_p}$ ⟨ two roots are real; two roots are imaginary

$\sqrt{2} < \left|\frac{kv_0}{\omega_p}\right|$ all four roots are real

$\frac{kv_0}{\omega_p} = \frac{\sqrt{3}}{2}$ $\omega_{imaginary} = \omega_p/2$ (maximum)

$\frac{\omega_p L}{v_0} > \frac{2\pi}{\sqrt{2}}$ minimum unstable length



Beam plasma instability (BL p. 110)

$$\varepsilon(\omega, k) = 1 - \frac{\omega_{pp}^2}{\omega^2} - \frac{\omega_{pb}^2}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v}_0)^2}$$

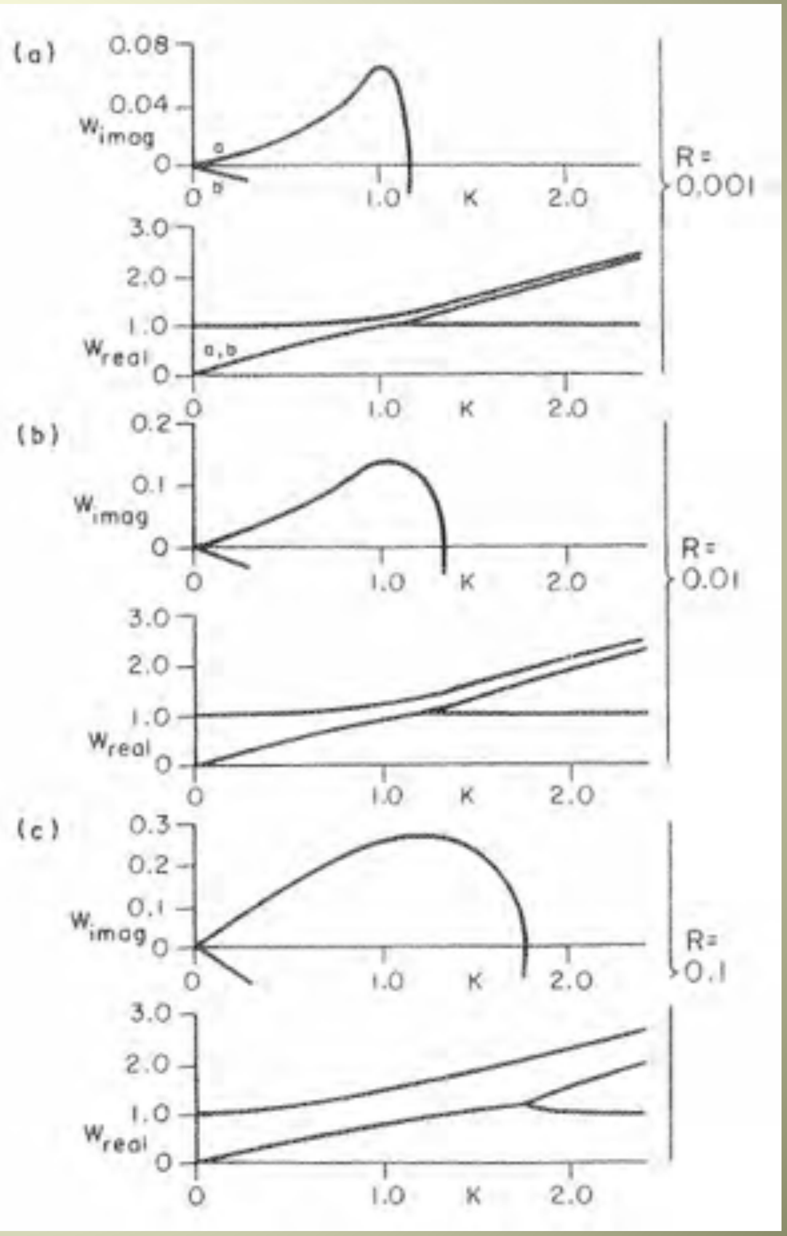
$$W \equiv \frac{\omega}{\omega_{pp}}, \quad K \equiv \frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega_{pp}}, \quad R = \left(\frac{\omega_{pb}}{\omega_{pp}}\right)^2$$

$$0 = F(W, K) = 1 - \frac{1}{W^2} - \frac{R}{(W-K)^2}$$

This instability can be tested by UPIC (homework) (5-12, p.119) phase-space plot?

weak beam

strong beam



Beam-cyclotron instability: Linear analysis (P.122)

$$\omega_{ci} \ll \omega_{pi} \ll \omega_{pe} \approx \omega_{ce},$$

$$\text{ion drift: } v_0, \text{ unmagnetized: } |\omega - \boldsymbol{k}\cdot \boldsymbol{v}_0| \gg \omega_{ci},$$

$$\text{unstable: } v_0 > v_{te} \text{ or } v_{ti},$$

hydrodynamic nonresonant instability

$$\frac{kv_{te}}{\omega_{ce}} < 1, \ |\omega - kv_0| > kv_{te}, \ \omega > k_{\parallel} v_{te},$$

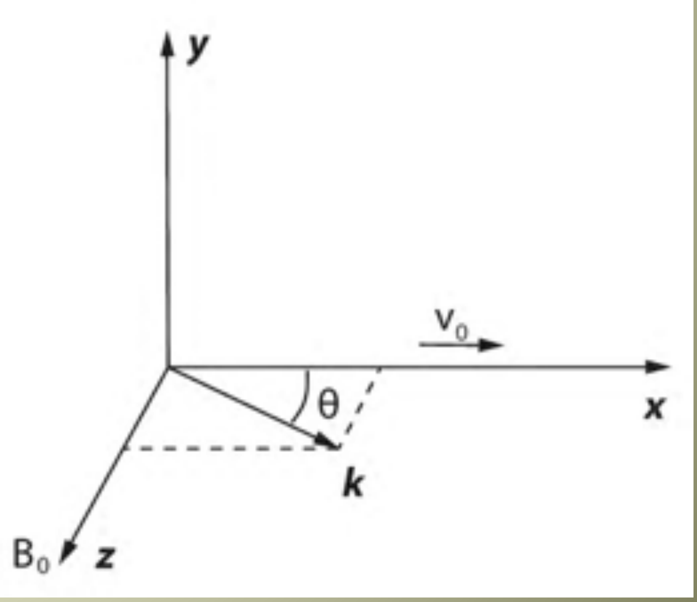
$$\frac{v_{te}}{v_0} < \frac{\omega_{ce}}{\omega_{uh}} \approx 1 \quad \text{for } \omega_{ce} \geq \omega_{pe}, \ \omega_{uh}^2 = \omega_{pe}^2 + \omega_{ce}^2$$

$$\frac{v_{ti}}{v_0} < \left(\frac{m_e}{m_i}\right)^{1/3} \left(\frac{\omega_{pe}}{\omega_{uh}}\right)^{4/3}$$

$$\text{for } T_e \approx T_i \ (m_e v_{te}^2 \approx m_i v_{ti}^2) \text{ and } \omega_{pe} \approx \omega_{ce}$$

$$\frac{v_{te}}{v_0} < \left(\frac{m_e}{m_i}\right)^{1/6} \left(\frac{\omega_{pe}}{\omega_{uh}}\right)^{4/3}$$

$$1 - \cos^2\theta \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2} - \sin^2\theta \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{(\omega - kv_0 \cos\theta)^2} = 0$$



Landau damping

B_0=E_0=0

$$\varepsilon(k,\omega) = 1 - \frac{1}{2}\left(\frac{S\omega_p}{kv_t}\right)^2 Z'\left(\frac{\omega}{\sqrt{2}kv_t}\right),\ \zeta = \frac{\omega}{\sqrt{2}kv_t}$$

in 1D simulation

$$\varepsilon(k,\omega) = 1 - \frac{1}{2}\left(\frac{S\omega_p}{k}\right)^2 \int_{-\infty}^{\infty} \frac{\partial \hat{f}_0(v_x)/\partial v_x}{(v_x - \omega/k)} dv_x$$

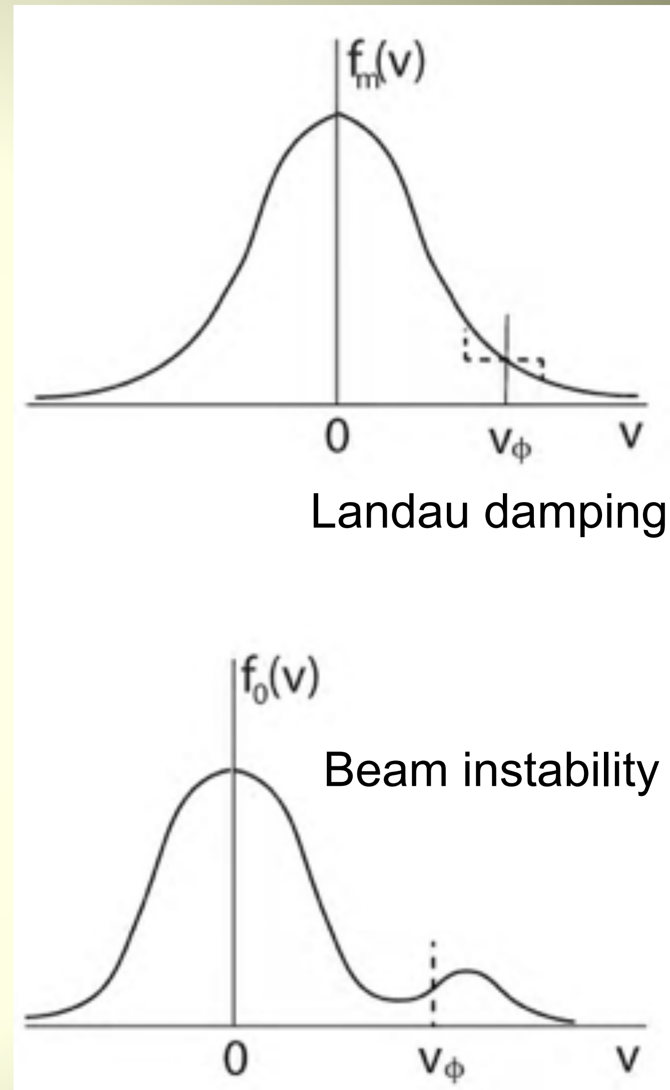
$$\hat{f}_0(v_x) = (m/2\pi KT)^{1/2} \exp(-mv_x^2/2KT)$$

$$\varepsilon(k,\omega_r + \omega_i) = 0$$

$$0 = \mathrm{Im}\,\varepsilon \approx 2\frac{\omega_{pc}^2}{\omega_r^3}\omega_i - \pi\frac{\omega_{ph}^2}{k|k|} f_0'\left(\frac{\omega_r}{k}\right)$$

$$\frac{\omega_i}{\omega_r} = -\left(\frac{\pi}{8}\right)^{1/2} \frac{\omega_{ph}^2}{\omega_{pc}^2}\left(\frac{\omega_r}{kv_t}\right)^3 e^{-\omega_r^2/2k^2v_t^2}$$

$$= -0.22\sqrt{\pi}\left(\frac{\omega_p}{kv_t}\right)^3 \exp\left(-1/2k^2\lambda_D^2\right) \text{ with } \omega_r = \omega_p$$



Magnetized ring-velocity distribution: Dory-Guest-Harris instability; Linear theory

$$B_0 \; (k_\perp \neq 0, k_\parallel = 0)$$

p=0: Maxwellian
p≠0: warm ring
p→∞: cold ring

$$dn = F_0^p(v_\perp) 2\pi v_\perp dv_\perp = \left| \int_{-\infty}^{\infty} dv_\parallel \; f_0(v_\perp, v_\parallel) \right| 2\pi v_\perp dv_\perp$$

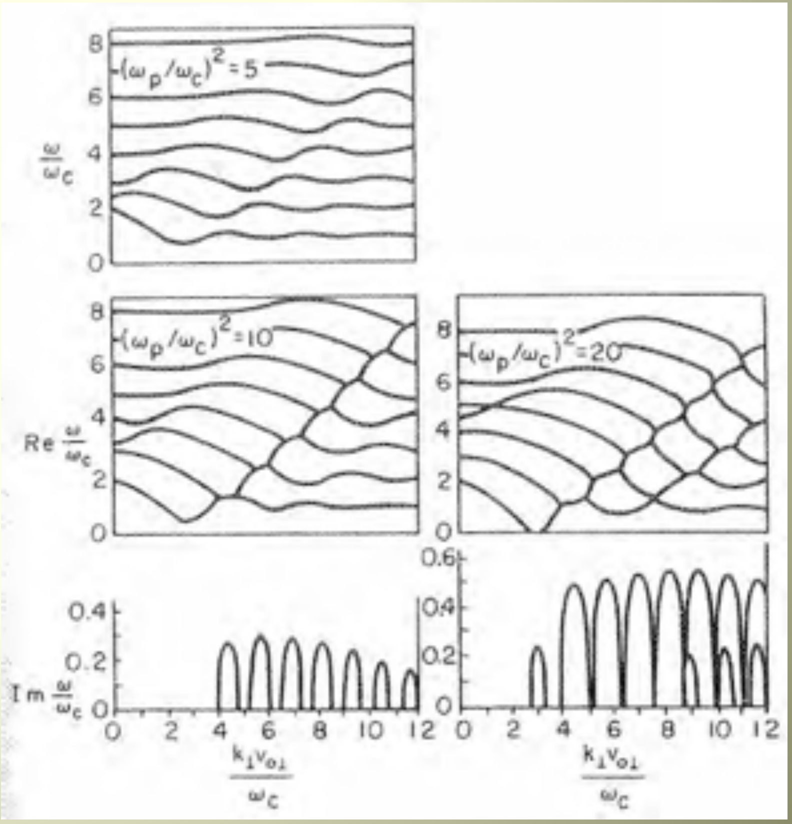
$$= \frac{1}{\pi \alpha_\perp^2 p!} \left(-\frac{v_\perp}{\alpha_\perp} \right)^{2p} \exp\left(-\frac{v_\perp^2}{\alpha_\perp^2} \right) 2\pi v_\perp dv_\perp$$

$$\frac{\partial f_1}{\partial t} + \boldsymbol{v} \cdot \nabla f_1 + \frac{q}{m} (\boldsymbol{v} \times \boldsymbol{B}_0) \cdot \frac{\partial f_1}{\partial \boldsymbol{v}} = -\frac{q}{m} (\boldsymbol{E}_1 + \boldsymbol{v} \times \boldsymbol{B}_1) \cdot \frac{\partial f_0}{\partial \boldsymbol{v}}$$

Bernstein waves

$$1 = \sum_s \frac{\omega_p^2}{\omega_c^2} \frac{2}{b} e^{-b} \sum_{n=1}^{\infty} \frac{I_n(b)}{(\omega / n\omega_c)^2 - 1}, \quad b = k_x^2 v_t^2 / 2\omega_c^2$$

$$I_n(b) \approx (b/2)^n / n!$$



Loading Maxwellian distribution f_0(v) BL p. 390

$$R_s(0 \to 1) = F(v) = \frac{\int_0^v \exp\left(-\frac{v^2}{2v_t^2}\right) dv}{\int_0^\infty \exp\left(-\frac{v^2}{2v_t^2}\right) dv}$$

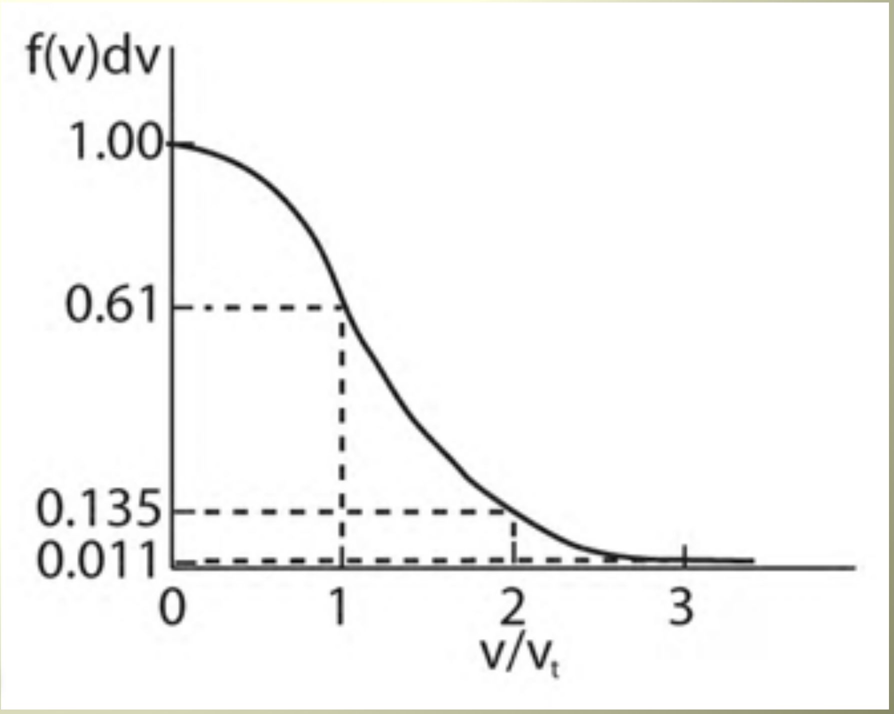
For a two-dimensional isotropic thermal distribution

$$v = \sqrt{v_x^2 + v_y^2} \text{ and } \theta = \tan^{-1} v_x / v_y \quad d\mathbf{v} = 2\pi v dv$$

$$v_s = v_t\sqrt{-2\ln R_s}$$

$$v_{s,\theta} = v_t\sqrt{-2\ln R_s}\cos 2\pi R_\theta$$

$$v_{s,\theta} = v_t\sqrt{-2\ln R_s}\sin 2\pi R_\theta$$

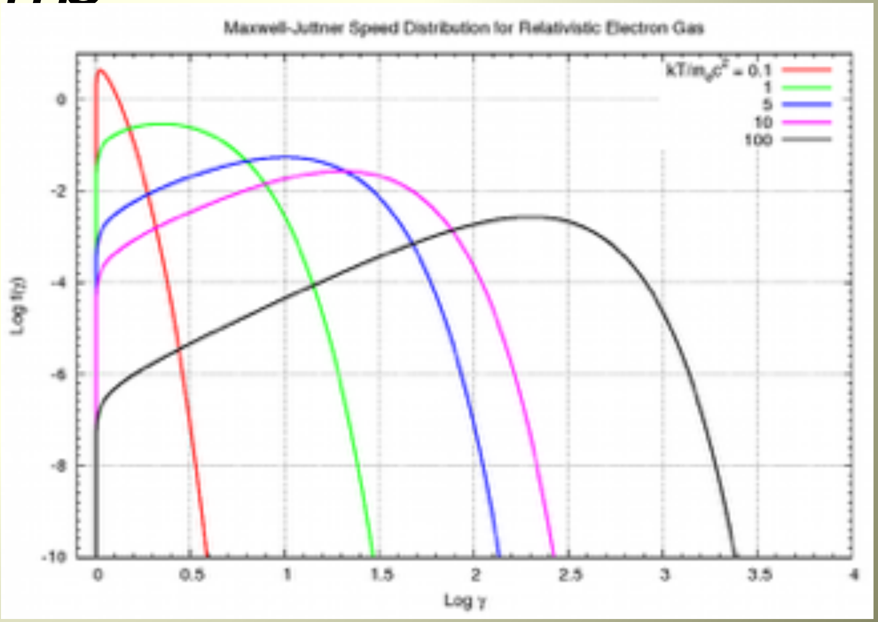


Relativistic Drift Maxwellian distribution

1-D velocity distribution

$$f(\gamma) = \frac{\gamma^2 \beta}{\theta K_2(1/\theta)} \exp\left(-\frac{\gamma}{\theta}\right)$$

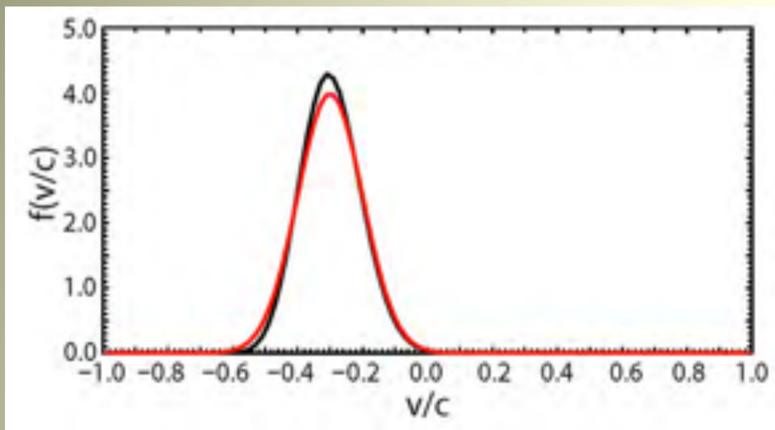
$$\beta = \frac{v}{c}, \gamma = \frac{1}{\sqrt{1-\beta^2}}, \theta = \frac{kT}{mc^2}$$



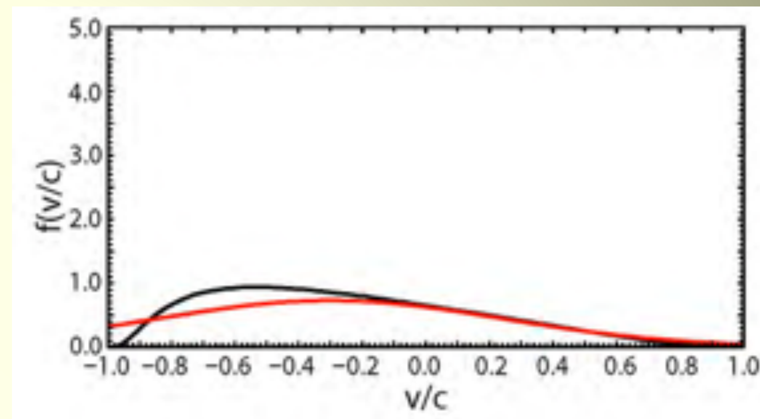
Examples of Jüttner distributions using synge.f

relativistic

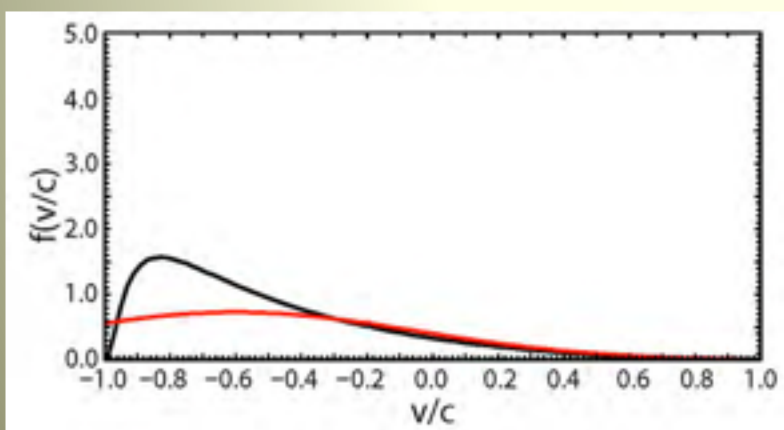
non-relativistic



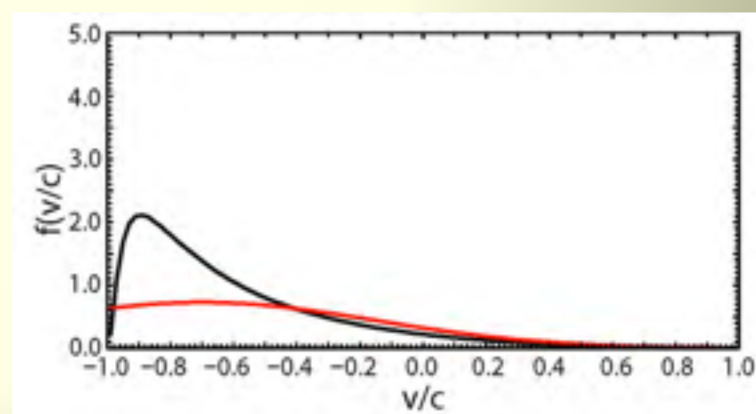
$\beta = -0.3c$, $\Theta = 0.01mc^2$



$\beta = -0.3c$, $\Theta = 0.3mc^2$

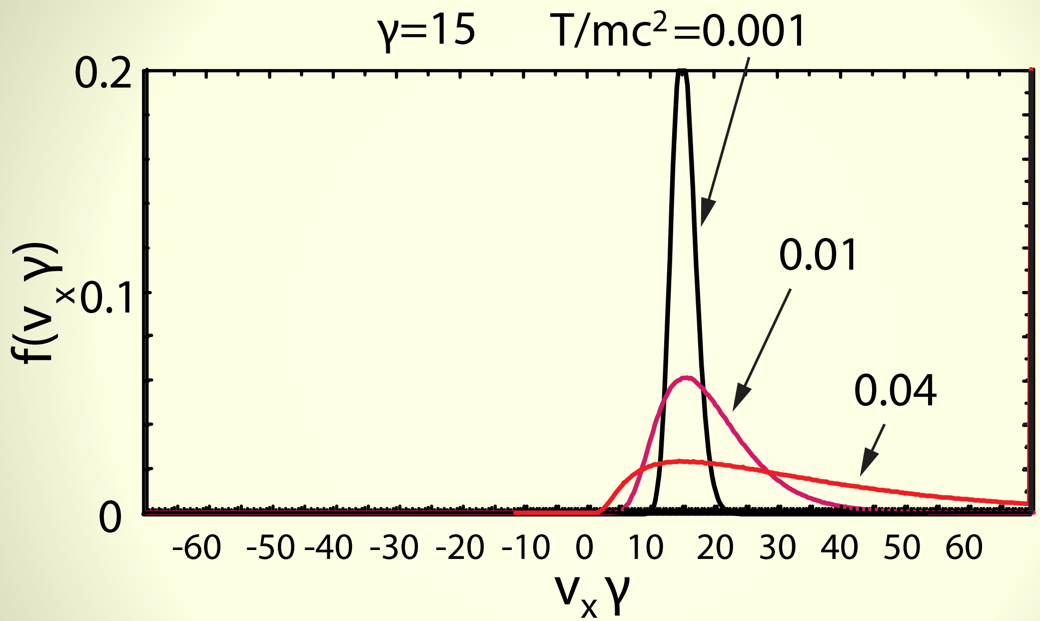


$\beta = -0.6c$, $\Theta = 0.3mc^2$



$\beta = -0.7c$, $\Theta = 0.3mc^2$

Relativistic drift Maxwellian distribution of jets different thermal temperature



Drifting relativistic Maxwellian distribution

$$\boldsymbol{p} = \gamma m \boldsymbol{v}, \quad \gamma = 1/(1-\boldsymbol{v}^2/c^2)^{1/2}, \quad E = (m^2 c^4 + c^2 \boldsymbol{p}^2)^{1/2}$$

drifting Jüttner/Synge distribution

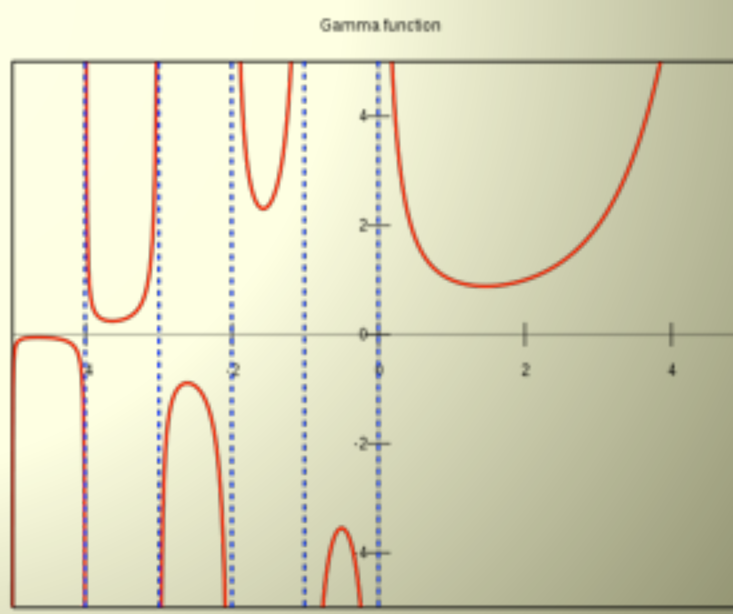
$$f_0 = \frac{\bar{N}}{4\pi m^2 c \Theta K_2(mc^2/\Theta)} \exp\left(-\frac{\bar{E}}{\Theta}\right),$$

$\bar{E}$: the particle energy in the rest frame, Θ :the temperature

After Lorentz transformation

$$E = \Gamma(E - c\beta p_y) = \int_0^\infty t^{E - c\beta p_y - 1} e^{-t}\, dt, \quad \beta = v_{0y}/c$$

$$f_0 = \frac{\bar{N}}{4\pi m^2 c \Theta K_2(mc^2/\Theta)} \exp\left(-\frac{\Gamma(E - c\beta p_y)}{\Theta}\right)$$



How to load drifting relativistic Maxwellian distribution

```
 program synge

*    test the drifting relativistic maxwellians

     real*4 beta,temp,vx,vy,vz,dvy,vycentre,fnonrel,pi
    1     ,vyarray(1000),df(1000)
     integer n,nparts,npoints,j
     open(unit=4,file='synge.dat',status='unknown')

     nparts=5000000
     npoints=100
     temp=.01
     beta=-0.26
```

```
*        set up the arrays

      dvy=2./real(npoints)
      do n=1,npoints+1
        df(n)=0.
        vyarray(n)=-1.+real(n-1)*dvy
      end do

      do n=1,nparts
        call initialise(beta,temp,vx,vy,vz)
        call locate(vyarray,npoints,vy,j)
        if(j.lt.1.or.j.gt.npoints)then
          write(*,*)'error, j=',j,' npoints=',npoints
          write(*,*)'vy=',vy
        end if
        df(j)=df(j)+1.
      end do
```

```
*         Write out the arrays and the velocity correctly centred
*         and compare with the analytic, nonrelativistic,
*         drifting Maxwellian

        pi=4.*atan(1.)
        write(4,1000)npoints
        do n=1,npoints
        vycentre=vyarray(n)+dvy/2.
        fnonrel=(2.*pi*temp)**(-.5)*exp(-(vycentre-beta)**2/2./temp)
        write(4,2000)vycentre,df(n)/dvy/real(nparts),fnonrel
        end do

 1000 format(1x,i5)
 2000 format(1x,3(e16.8,1x))
        stop
        end
```

```
******************************************************************

        subroutine initialise(beta,temp,vx,vy,vz)
        real*4 beta,gdrift,temp,theta,vx,vy,vz,one,r1,r2,r3,r4
     1      ,eta,etap,etapp,zetap,gamma,ran1,radius
     2      ,pxp,pyp,pzp,gammap

        integer idum

*           Model the drifting relativistic Maxwellians
*           for the relativsitic Harris sheet
*           INPUT
*             beta: the (modulus of the) drift speed of each
*                 component with respect to the "lab" frame
*             temp: the temperature T in units of mc^2/kB
*                 In the co-drifting frames the temperature
*                 is \Theta=\Gamma*temp - see notes in KS03

*     Use the methods described by Pozdnyakov and Sobol (1983) for
*     the relativistic Maxwellian

       data idum/1234/
```

```
 one=1.d0
      gdrift=1./sqrt(one-beta**2)
      theta=temp*gdrift

*         Find eta=p/mc

      if(theta.lt.0.29)then
 1        r1=ran1(idum)
          r2=ran1(idum)
          zetap=-1.5*log(r1)
          if(
     1        r2**2.lt.0.51*(1.+theta*zetap)**2*zetap*(2.+theta*zetap)*r1
     2      )then
            eta=sqrt(theta*zetap*(2.+theta*zetap))
          else
            go to 1
          end if
```

```
    else
 2       r1=ran1(idum)
        r2=ran1(idum)
        r3=ran1(idum)
        r4=ran1(idum)
        etap=-theta*log(r1*r2*r3)
        etapp=-theta*log(r1*r2*r3*r4)
        if(etapp**2-etap**2.gt.1.)then
          eta=etap
        else
          go to 2
        end if
     end if

*       Draw a random direction
```

```
 3    r1=2.*(ran1(idum)-.5)
      r2=2.*(ran1(idum)-.5)
      r3=2.*(ran1(idum)-.5)
      radius=sqrt(r1**2+r2**2+r3**2)
      if(radius.lt.1.)then
        pxp=eta*r1/radius
        pyp=eta*r2/radius
        pzp=eta*r3/radius
      else
        go to 3
      end if

*       Now compute the lab frame output quantities

      gammap=sqrt(one+eta**2)
      gamma=gdrift*(gammap+beta*pyp)
      vx=pxp/gamma
      vz=pzp/gamma
      vy=gdrift*(pyp+beta*gammap)/gamma
      return
      end
```