Lecture 1:

Relativistic Astrophysics and Magnetohydrodynamics

Yosuke Mizuno
ITP, Goethe University Frankfurt

Spacial lecture “GRMHD”, August 13th-17th, USP-IAG, Sao Paulo, Brazil

Relativistic Regime

- Kinetic energy >> rest-mass energy
  - Fluid velocity ~ light speed (Lorentz factor γ>> 1)
  - Relativistic jets/ejecta/wind/blast waves (shocks) in AGNs, GRBs, Pulsars
- Thermal energy >> rest-mass energy
  - Plasma temperature >> ion rest mass energy ($p/\rho c^2 \sim k_B T/mc^2 >> 1$)
  - GRBs, magnetar flare?, Pulsar wind nebulae
- Magnetic energy >> rest-mass energy
  - Magnetization parameter $\sigma >> 1$
  - σ = Poyniting to kinetic energy ratio = $B^2/4\pi\rho c^2\gamma^2$
  - Pulsars magnetosphere, Magnetars
- Gravitational energy >> rest-mass energy
  - $GMm/rmc^2 = r_g/r > 1$
  - Black hole, Neutron star
- Radiation energy >> rest-mass energy
  - $E'_r/\rho c^2 >>1$
  - Supercritical accretion flow

Applications of Relativistic Astrophysics

- Black Holes:
  - high, low accretion rate AGN
  - tidal disruption event
  - X-ray binaries
  - long-soft GRBs
  - BH-BH merger for GW sources
- Neutron stars:
  - pulsar magnetosphere
  - core-collapse supernova
  - short-hard GRBs
  - NS-NS merger for GW sources
- Jets/relativistic wind:
  - extra-galactic jets/outflows
  - pulsar jet/wind
  - microquasars
  - gamma-ray bursts
- Laboratory physics:
  - relativistic heavy-ion collision
  - plasma laboratory experiments

Relativistic Jets

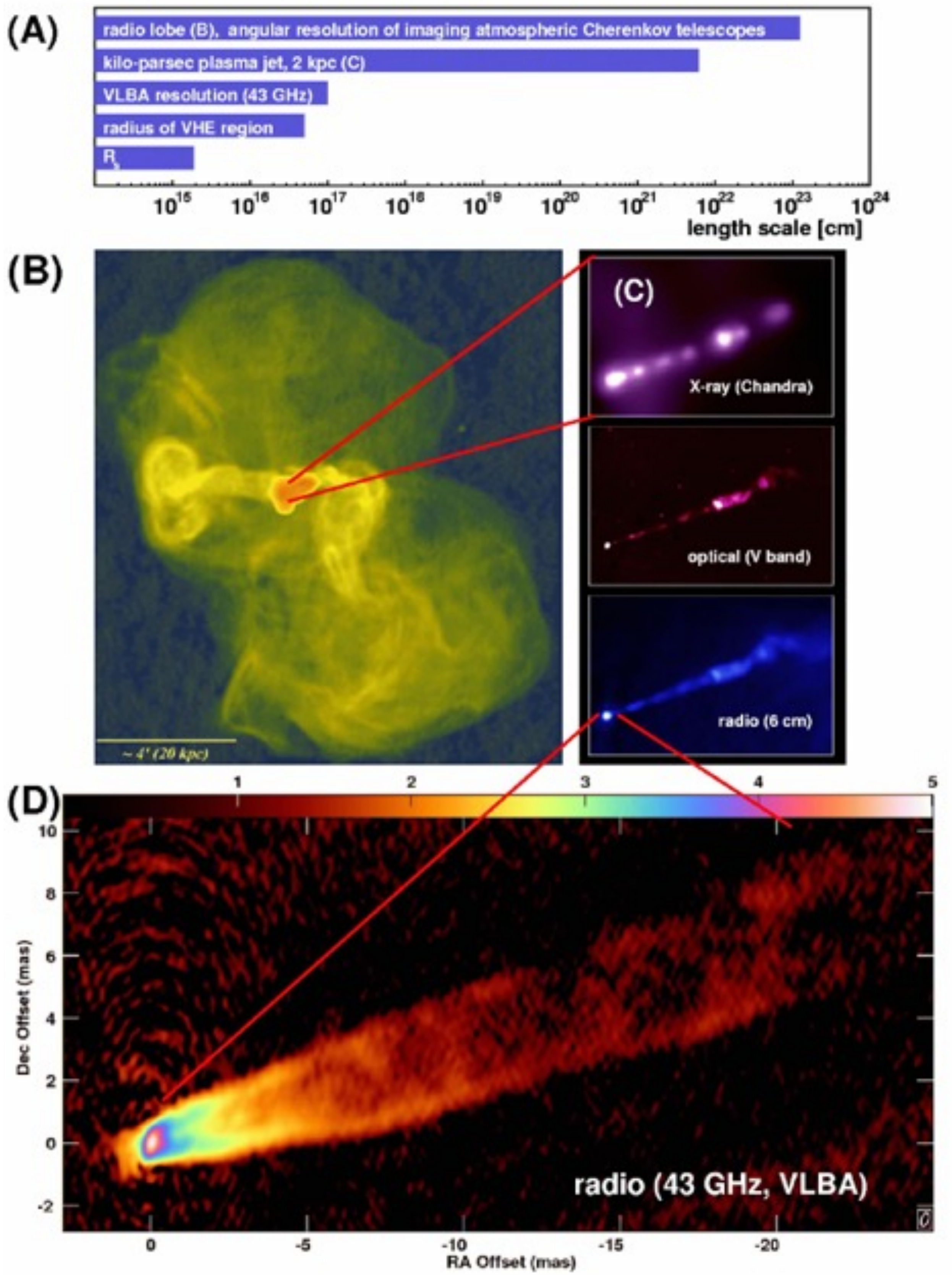
- *Relativistic jets: outflow of highly collimated plasma*
  - Microquasars, Active Galactic Nuclei, Gamma-Ray Bursts, Jet velocity ～c
  - Generic systems: Compact object （White Dwarf, Neutron Star, Black Hole） + Accretion Disk

- *Key Issues of Relativistic Jets*
  - Acceleration & Collimation
  - Propagation & Stability

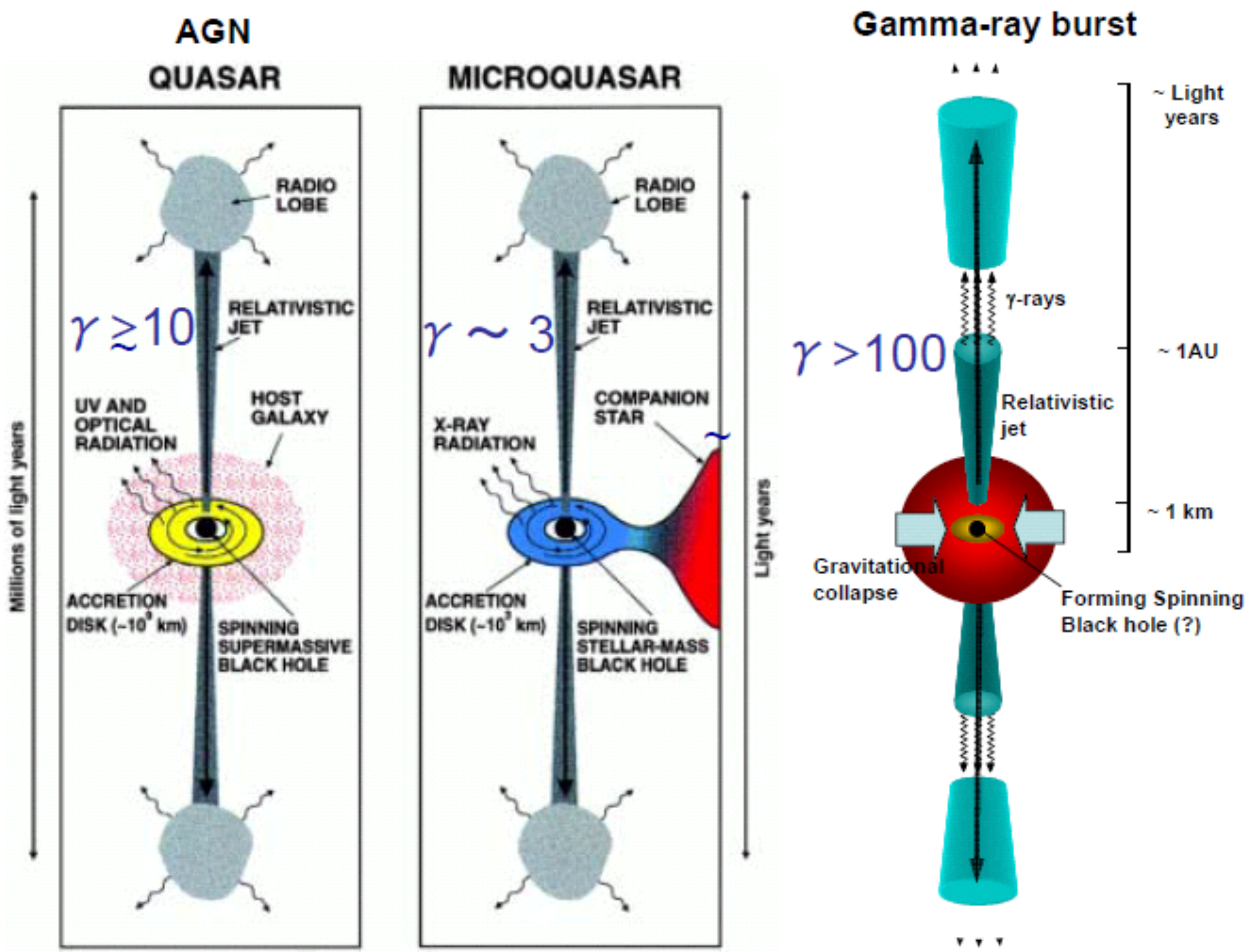
- *Modeling for Jet Production*
  - Magnetohydrodynamics (MHD)
  - Relativity (SR or GR)

- *Modeling of Jet Emission*
  - Particle Acceleration
  - Radiation mechanism

Radio observation of M87 jet



Relativistic Jets in Universe

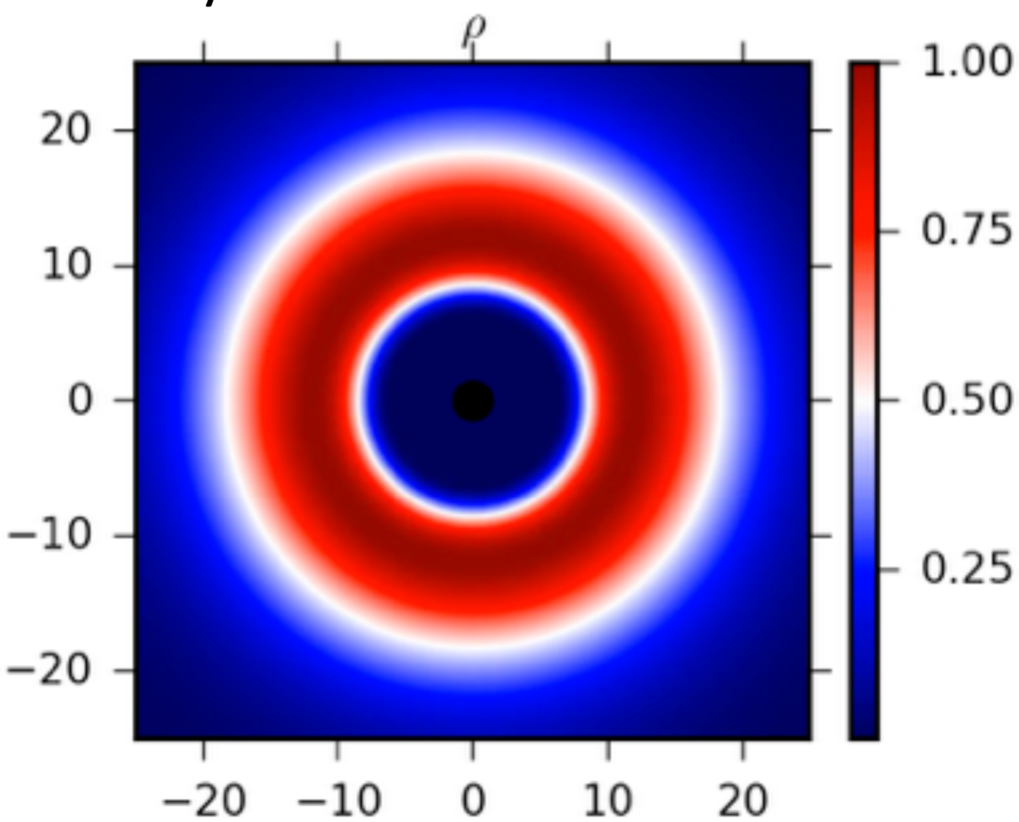


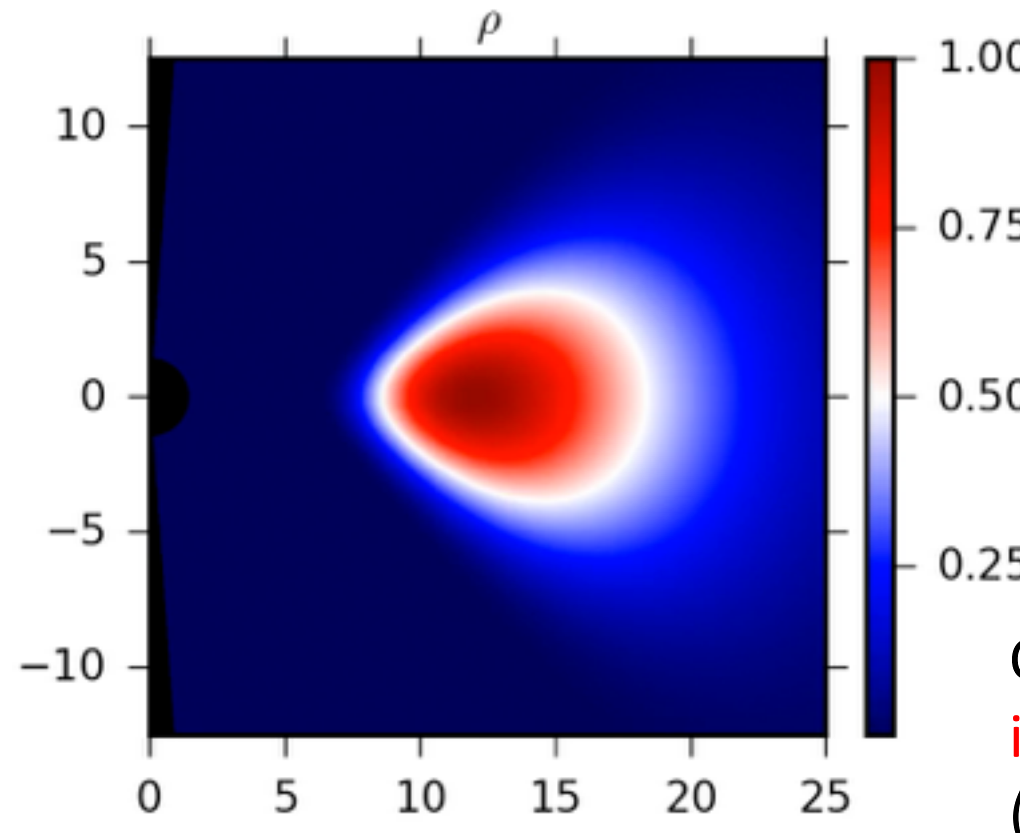
Mirabel & Rodoriguez 1998

Plasma Dynamics vicinity of BH and Shadow

density

Porth et al. (2017)

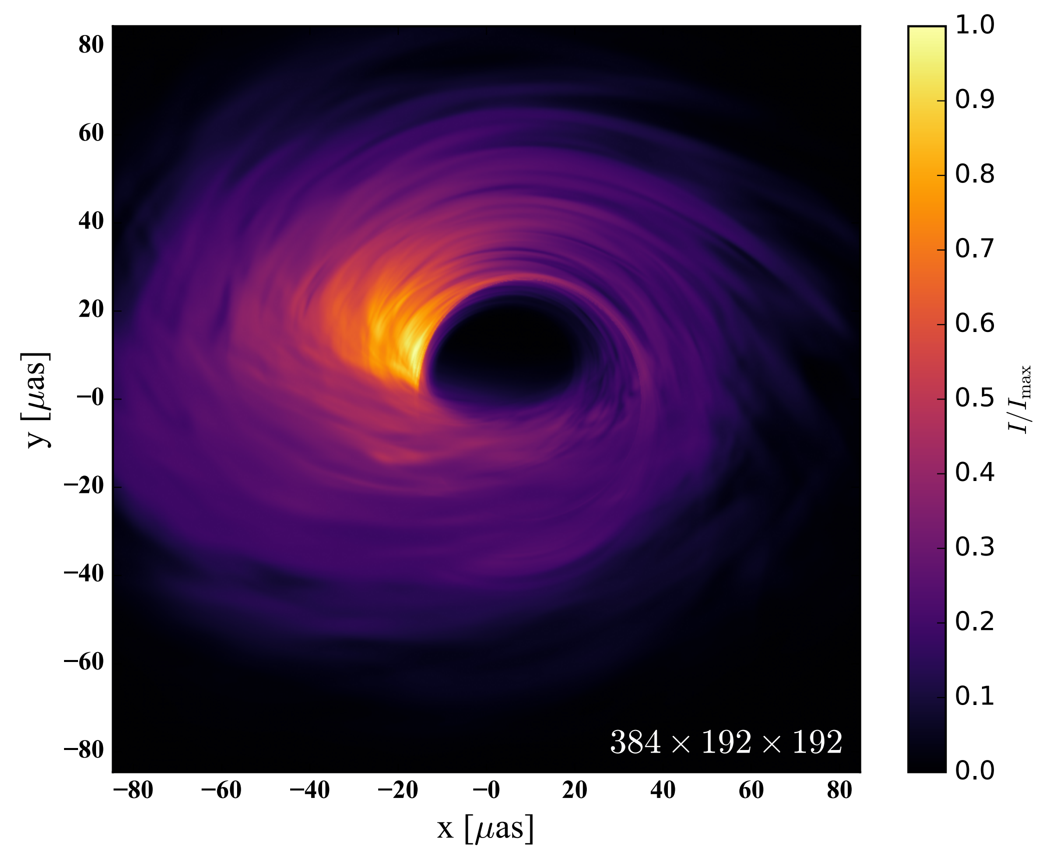




- Initial: Accretion torus + weak single magnetic field loop
- Inside torus becomes turbulent by MRI
- Poynting flux dominated jet is developed near the axis

Calculated Radiation image by GRRT code (Thermal synchrotron total intensity)

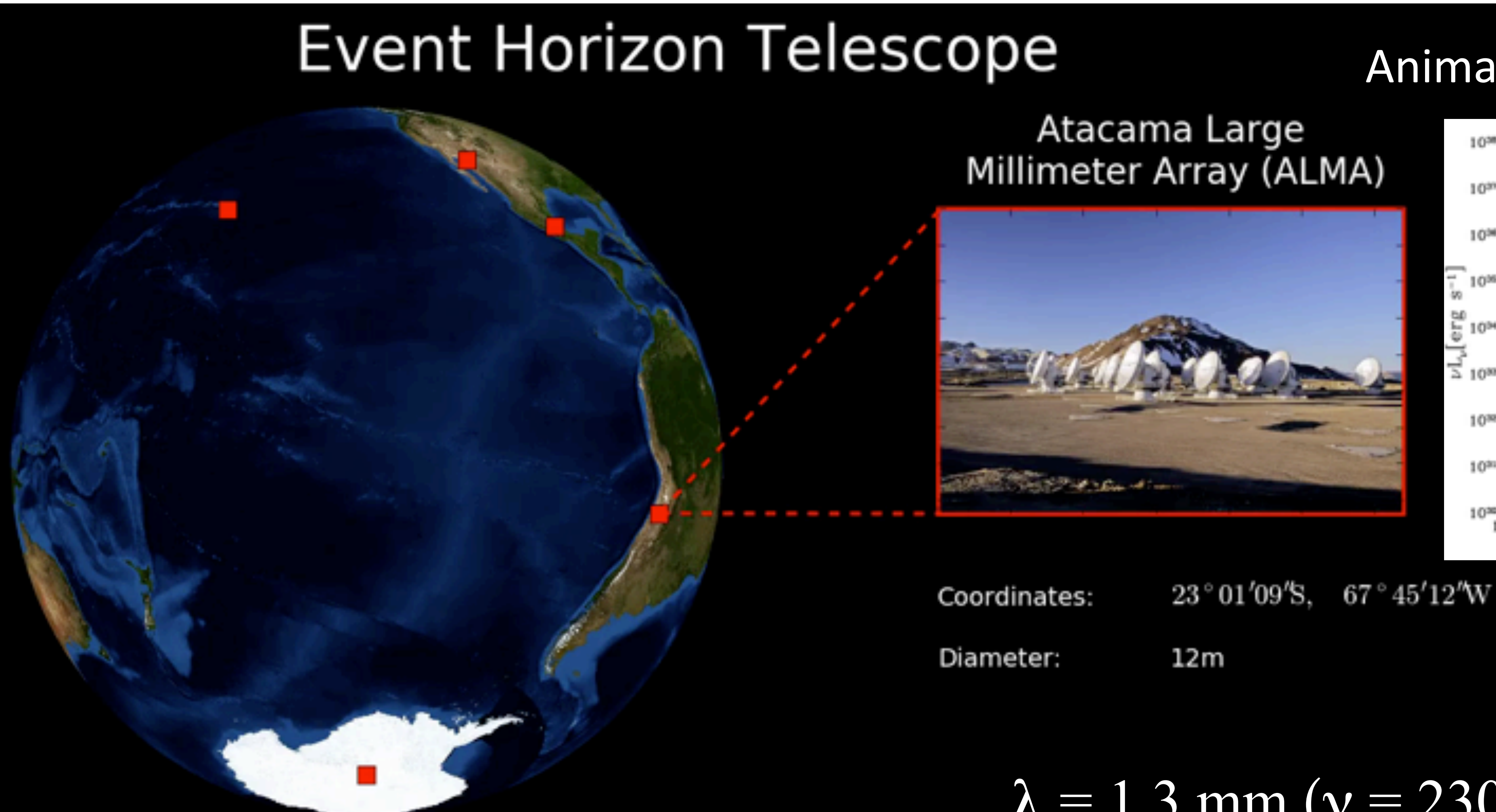
- We can obtain BH shadow image, spectrum, light curve (+ polarization) via 3D GRMHD simulations



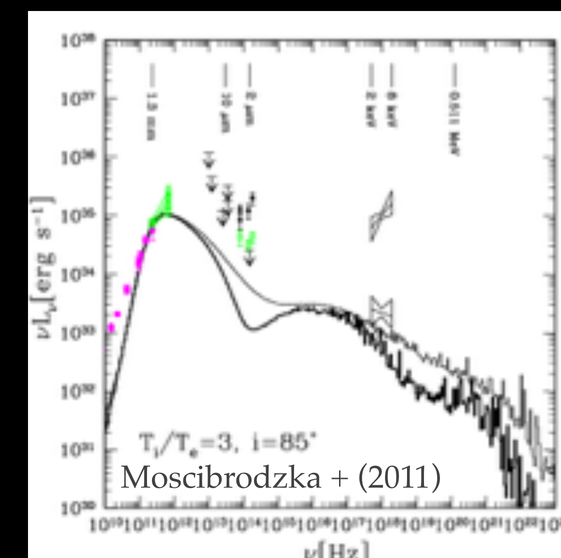
Event Horizon Telescope

International collaboration project of Very Long Baseline Interferometry (VLBI) at mm (sub-mm) wavelength

Event Horizon Telescope



Animation: C. Fromm



Create a virtual radio telescope the size of the earth, using the shortest wavelength

$\lambda = 1.3$ mm ($\nu = 230$ GHz)

$D \sim 10{,}000$ km

$\Rightarrow \lambda/D \sim 25\ \mu$as

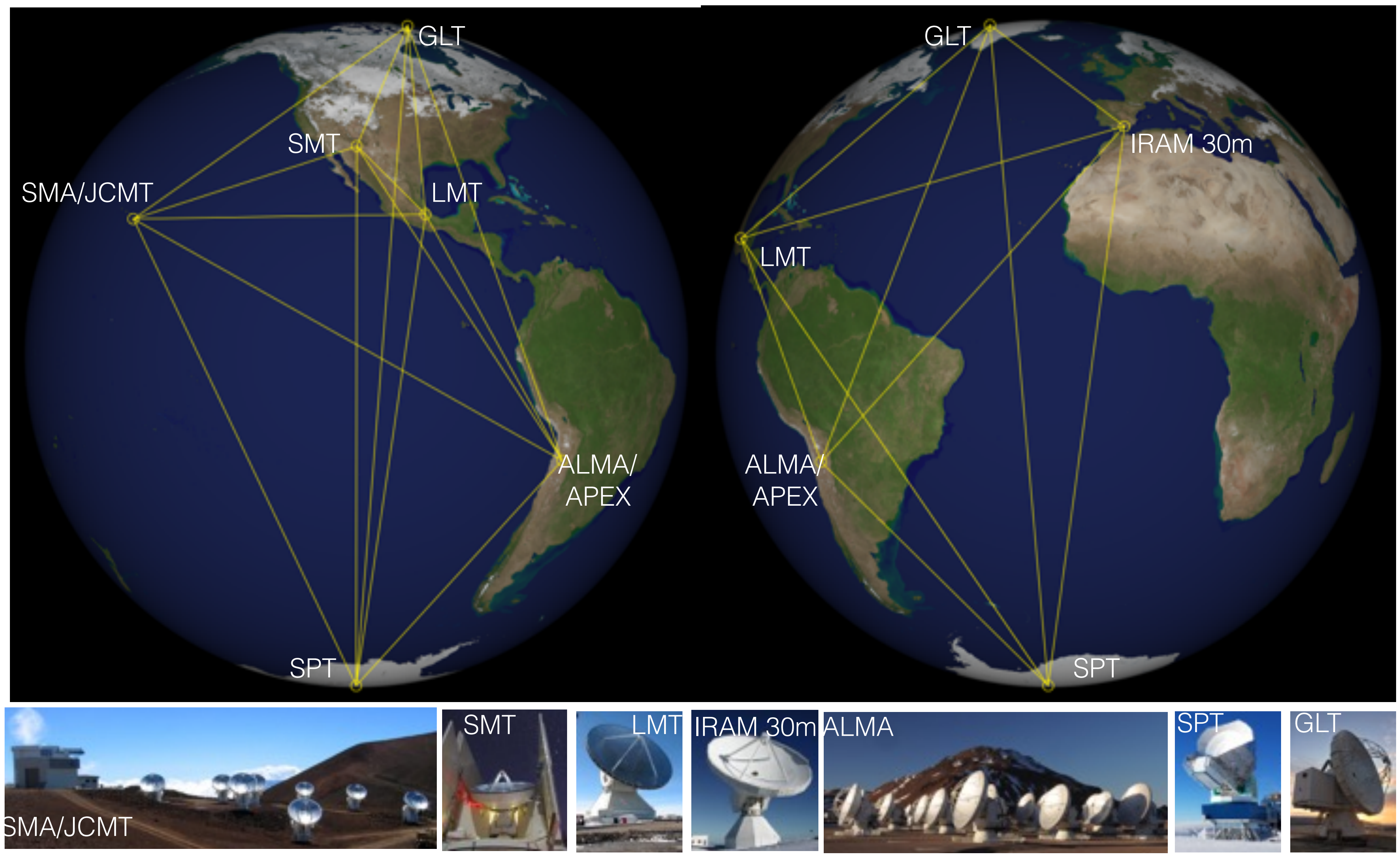
Two main targets: Sgr A* & M87

Event Horizon Telescope in 2017

- Atacama Large Millimeter Array (ALMA), Chile
- ALMA Pathfinder Experiment (APEX), Chile
- James Clerk Maxwell Telescope (JCMT), Hawaii
- Large Millimeter Telescope (LMT), Mexico
- IRAM 30-meter Telescope, Spain
- South Pole Telescope (SPT), South Pole
- Submillimeter Array (SMA), Hawaii
- Submillimeter Telescope (SMT), Arizona

M. Johnson/SAO

Event Horizon Telescope in 2018



D. Marrone/UofA

Fluid Dynamics

- Fluid dynamics deals with the behaviour of matter in the large (average quantities per unit volume), on a macroscopic scale large compared with the distance between molecules, $l \gg d_0 \sim 3\text{-}4 \times 10^{-8}$ cm, not taking into account the molecular structure of fluids.

- Macroscopic behaviour of fluids assumed to be continuous in structure, and physical quantities such as mass, density, or momentum contained within a given small volume are regarded as uniformly spread over that volume.

- The quantities that characterize a fluid (in the continuum limit) are functions of time and position:

$$\rho \; : \; (t, \vec{r}) \in \mathbb{R}^4 \rightarrow \rho(t, \vec{r}) \in \mathbb{R}$$ density (scalar field)

$$\vec{v} \; : \; (t, \vec{r}) \in \mathbb{R}^4 \rightarrow \vec{v}(t, \vec{r}) \in \mathbb{R}^3$$ velocity (vector field)

$$\Pi \; : \; (t, \vec{r}) \in \mathbb{R}^4 \rightarrow \Pi(t, \vec{r}) \in \mathbb{R}^9$$ pressure tensor (tensor field)

Fluid Approach to Plasmas

- Fluid approach describes bulk properties of plasma. We do not attempt to solve unique trajectories of all particles in plasma. This simplification works very well for majority of plasma.
- Fluid theory follows directly from moments of the Boltzmann equation.
- Each of moments of Boltzmann (Vlasov) equation is a transport equation describing the dynamics of a quantity associated with a given power of ***v***

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\boldsymbol{u}) = 0$$

Continuity of mass or charge transport

$$mn\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\right] = qn(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) - \nabla \cdot \boldsymbol{P} + \boldsymbol{P}_{ij}$$

Momentum transport

$$\frac{\partial}{\partial t}\left[n\frac{1}{2}mu^2\right] + \nabla \cdot \left[n\frac{1}{2}m\langle u^2\boldsymbol{u}\rangle\right] - nq\langle \boldsymbol{E} \cdot \boldsymbol{u}\rangle = \frac{m}{2}\int u^2 \left(\frac{\partial f}{\partial t}\right)_{coll} d\boldsymbol{u}$$

Energy transport

Single-Fluid Theory: MHD

- Under certain circumstances, appropriate to consider entire plasma as a single fluid.
- Do not have any difference between ions and electrons.
- Approach is called *magnetohydrodynamics (MHD).*

- General method for modeling highly conductive fluids, including low-density astrophysical plasmas.

- Single-fluid approach appropriate when dealing with slowly varying conditions.

- MHD is useful when plasma is highly ionized and electrons and ions are forced to act in unison, either because of frequent collisions or by the action of a strong external magnetic field.

Applicability of Hydrodynamic Approximation

- To apply hydrodynamic approximation, we need the condition:
  - Spatial scale >> mean free path
  - Time scale >> collision time
- These are not necessarily satisfied in many astrophysical plasmas
  - E.g., solar corona, galactic halo, cluster of galaxies etc.
- But in magnetized plasmas, the effective mean free path is given by the ion Larmor radius.
- Hence if the size of phenomenon is much larger than the ion Larmor radius, hydrodynamic approximation can be used.

Applicability of MHD Approximation

- Magnetohydrodynamics (MHD) describe macroscopic behavior of plasmas if
  - Spatial scale >> ion Larmor radius
  - Time scale >> ion Larmor period
- MHD can not treat
  - Particle acceleration
  - Origin of resistivity
  - Electromagnetic waves
  - etc

Fluid Motion

- The motion of fluid is described by a vector velocity field $\boldsymbol{v}(\boldsymbol{r})$, (which is mean velocity of all individual particles which make up the fluid at $\boldsymbol{r}$ and particle density $\boldsymbol{n}(\boldsymbol{r})$.
- We discuss the motion of fluid of a *single type* of particle of mass/charge, m/q, so charge and mass density are qn and mn
- The particle conservation equation (continuity equation):

$$\frac{\partial}{\partial t}n + \nabla \cdot (n\boldsymbol{v}) = 0$$

- Expand the $\nabla \cdot$ to get: $\frac{\partial}{\partial t}n + (\boldsymbol{v} \cdot \nabla)n + n\nabla \cdot \boldsymbol{v} = 0$
- Significance is that first two terms are *convective derivative* of n

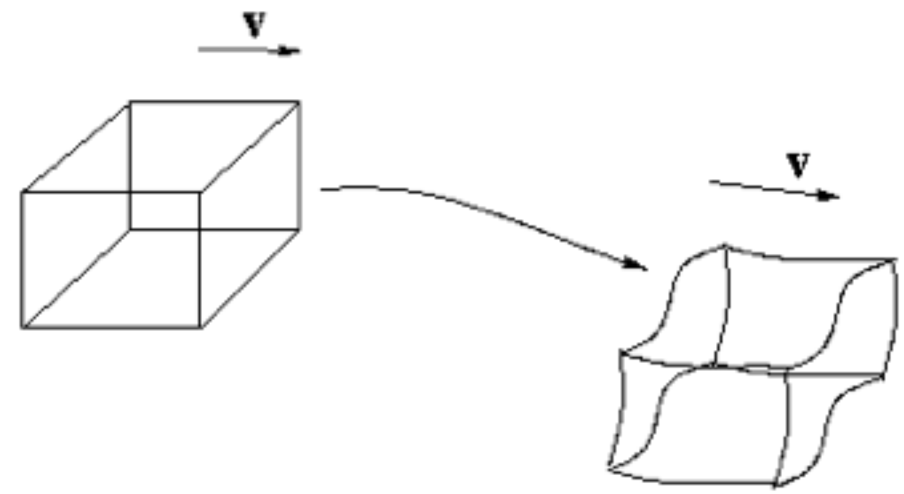
$$\frac{D}{Dt} \equiv \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla$$

- So continuity equation can be written: $\frac{D}{Dt}n = -n\nabla \cdot \boldsymbol{v}$

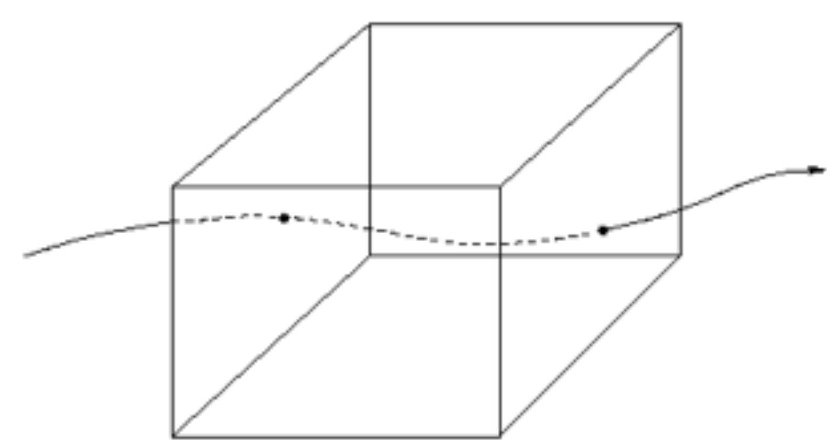
Lagrangian & Eulerian Viewpoint

- Lagrangian: sit on a fluid element and move with it as fluid moves

- Eulerian: sit at a fixed point in space and watch fluid move through your volume element: identity of fluid in volume continually changing
  - $\partial/\partial t$: rate of change at fixed point (Euler)
  - $D/Dt \equiv \partial/\partial t + \boldsymbol{v} \cdot \nabla$: rate of change at *moving* point (Lagrange)
  - $\boldsymbol{v} \cdot \nabla$: change due to motion



Lagrangian viewpoint



Eulerian viewpoint

Single-Fluid Equations for Fully Ionized Plasma

- Can combine multiple-fluid equations into a set of equations for a single fluid.

- Assuming two-specials plasma of electrons and ions ($j = e$ or i):

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \boldsymbol{v}_j) = 0$$

$$m_j n_j \left[\frac{\partial \boldsymbol{v}_j}{\partial t} + (\boldsymbol{v}_j \cdot \nabla) \boldsymbol{v}_j \right] = -\nabla \cdot \boldsymbol{P}_j + q_j n_j (\boldsymbol{E} + \boldsymbol{v_j} \times \boldsymbol{B}) + P_{ij}$$

- For a fully ionized two-species plasma, total momentum must be conserved:

$$P_{ei} = -P_{ie}$$

- As $m_\mathrm{i} >> m_\mathrm{e}$ the time-scales in continuity and momentum equations for ions and electrons are very different. The characteristic frequencies of a plasma, such as plasma frequency or cyclotron frequency are much larger for electrons.

Single-Fluid Equations for Fully Ionized Plasma

- When plasma phenomena are large-scale ($L >> \lambda_D$) and have relatively low frequencies ($\omega << \omega_{plasma}$ and $\omega << \omega_{cyclotron}$), on average plasma is electrically neutral ($n_i \sim n_e$). Independent motion of electrons and ions can then be neglected.
- Can therefore treat plasma as single conducting fluid, whose inertia is provided by mass of ions.
- Governing equations are obtained by combining two equations (electron +ions)
- First, define macroscopic parameters of plasma fluid:

$$\rho_m = n_e m_e + n_i m_i$$ Mass density

$$\rho_e = n_e q_e + n_i q_i$$ Charge density

$$\boldsymbol{J} = n_e q_e v_e + n_i q_i v_i = n_e q_e (v_e - v_i)$$ Electric current

$$\boldsymbol{v} = (n_e m_e \boldsymbol{v}_e + n_i m_i \boldsymbol{v}_i)/\rho_m$$ Center of Mass Velocity

$$\boldsymbol{P} = \boldsymbol{P}_e + \boldsymbol{P}_i$$ Total pressure tensor

MHD Mass and Charge Conservation

- Using continuity eq: $\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \boldsymbol{v}_j) = 0$
- Multiply by q_i and q_e and add continuity equations to get:

$$\boxed{\frac{\partial \rho_e}{\partial t} + \nabla \cdot (\boldsymbol{J}) = 0}$$

Charge conservation

- where J is the electric current density: $\boldsymbol{J} = n_e q_e \boldsymbol{v}_e + n_i q_i \boldsymbol{v}_i$ and the electric charge: $\rho_e = n_e q_e + n_i q_i$

- Multiply eq by m_i and m_e,

$$\boxed{\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \boldsymbol{v}) = 0}$$

Mass conservation / continuity equation

- where $\rho_m = n_e m_e + n_i m_i$ is the single-fluid mass density and ***v*** is the fluid mass velocity $\boldsymbol{v} = (n_e m_e \boldsymbol{v}_e + n_i m_i \boldsymbol{v}_i)/\rho_m$

MHD Equation of Motion

- Equation of motion for bulk plasma can be obtained by adding individual momentum transport equations for ions and electrons.
- LHS of momentum transport eq: $m_j n_j \left[\frac{\partial \boldsymbol{v}_j}{\partial t} + (\boldsymbol{v}_j \cdot \nabla)\boldsymbol{v}_j \right]$
- Difficulty is that convective term is *non-linear.*
- But note that since $m_e << m_i$ contribution of electron momentum is much less than that from ion. So we ignore it in equation
- Approximation: Center of mass velocity is ion velocity: $\boldsymbol{v} \simeq \boldsymbol{v}_i$
- LHS of momentum transport eq:

$$m_j n_j \left[\frac{\partial \boldsymbol{v}_j}{\partial t} + (\boldsymbol{v}_j \cdot \nabla)\boldsymbol{v}_j \right] \simeq \rho_m \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right]$$

MHD Equation of Motion

- RHS of momentum transport eq :

$$-\nabla \cdot (\boldsymbol{P}_e + \boldsymbol{P}_i) + (n_e q_e + n_i q_i)\boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}$$

- In general, second term (Electric body force) is much smaller than $\boldsymbol{J} \times \boldsymbol{B}$ term. So we ignored.
- Therefore, LHS+RHS:

$$\rho_m \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right] = -\nabla \cdot \boldsymbol{P} + \boldsymbol{J} \times \boldsymbol{B}$$

Equation of motion

- For an isotropic plasma, $\nabla \cdot \boldsymbol{P} = \nabla p$ where total pressure is $p = p_e + p_i$ and

$$\rho_m \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right] = -\nabla p + \boldsymbol{J} \times \boldsymbol{B}$$

Equation of motion

Generalized Ohm's Law

- The final single-fluid MHD equation describes the variation of current density $\boldsymbol{J}$.
- Consider the momentum equations for electron and ions:

$$m_j n_j \left[\frac{\partial \boldsymbol{v}_j}{\partial t} + (\boldsymbol{v}_j \cdot \nabla)\boldsymbol{v}_j\right] = -\nabla \cdot \boldsymbol{P}_j + q_j n_j(\boldsymbol{E} + \boldsymbol{v_j} \times \boldsymbol{B}) + P_{ij}$$

- Multiple electron equation by q_e/m_e and ion equation by q_i/m_i and add:

$$\begin{aligned}\frac{\partial \boldsymbol{J}}{\partial t} &= -\frac{q_e}{m_e}\nabla \cdot \boldsymbol{P}_e - \frac{q_i}{m_i}\nabla \cdot \boldsymbol{P}_i \\ &+ \left(\frac{n_e q_e^2}{m_e} + \frac{n_i q_i^2}{m_i}\right)\boldsymbol{E} \\ &+ \left(\frac{n_e q_e^2}{m_e}\boldsymbol{v}_e + \frac{n_i q_i^2}{m_i}\boldsymbol{v}_i\right) \times \boldsymbol{B} \\ &+ \frac{q_e}{m_e}\boldsymbol{P}_{ei} + \frac{q_i}{m_i}\boldsymbol{P}_{ie}\end{aligned}$$

(We ignore second term of LHS as we dealing with small perturbation)

Generalized Ohm's Law

- For an electrically neutral plasma $|q_e n_e| \approx |q_i n_i|$ and using $\boldsymbol{J} = n_e q_e \boldsymbol{v}_e + n_i q_i \boldsymbol{v}_i$ and $\boldsymbol{v} = (n_e m_e \boldsymbol{v}_e + n_i m_i \boldsymbol{v}_i)/\rho_m$ we can write

$$\begin{aligned}\frac{\partial \boldsymbol{J}}{\partial t} &= -\frac{q_e}{m_e}\nabla \cdot \boldsymbol{P}_e - \frac{q_i}{m_i}\nabla \cdot \boldsymbol{P}_i \\ &+ \left(\frac{n_e q_e^2}{m_e} + \frac{n_i q_i^2}{m_i}\right)(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \\ &+ \left(\frac{q_e}{m_e} + \frac{q_i}{m_i}\right)(\boldsymbol{J} \times \boldsymbol{B}) \\ &+ \left(\frac{q_e}{m_e} - \frac{q_i}{m_i}\right)\boldsymbol{P}_{ei}\end{aligned}$$

- As $m_e \ll m_i \rightarrow q_e/m_e \gg q_i/m_i$ and $n_e q_e^2/m_e \gg n_i q_i^2/m_i$. In thermal equilibrium, kinetic pressures of electrons is similar to ion pressure ($P_\mathrm{e} \sim P_\mathrm{i}$)

$$\frac{\partial \boldsymbol{J}}{\partial t} = -\frac{q_e}{m_e}\nabla \cdot \boldsymbol{P}_e + \frac{n_e q_e^2}{m_e}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) + \frac{q_e}{m_e}(\boldsymbol{J} \times \boldsymbol{B}) + \frac{q_e}{m_e}\boldsymbol{P}_{ei}$$

Generalized Ohm's Law

- The collisional term can be written: $\boldsymbol{P}_{ei} = \eta q^2 n_e^2 (\boldsymbol{v}_i - \boldsymbol{v}_e)$
  where η is the specific resistivity, q^2 relates to fact that collisions result from Coulomb force between ions (q_i) and electrons (q_e) and total momentum transferred to electrons in an elastic collision with an ion is $\boldsymbol{v}_\mathrm{i} - \boldsymbol{v}_\mathrm{e}$.

- Now q_i= - q_e and $n_\mathrm{e} = n_\mathrm{i}$ and $\boldsymbol{J}$=$n_\mathrm{e} q_\mathrm{e}(\boldsymbol{v}_\mathrm{e}$-$\boldsymbol{v}_\mathrm{i})$, => $\boldsymbol{P}_{ei} = -n_e q_e \eta \boldsymbol{J}$
- The equation can be written as

$$\frac{\partial \boldsymbol{J}}{\partial t} = -\frac{q_e}{m_e} \nabla \cdot \boldsymbol{P}_e + \frac{n_e q_e^2}{m_e} (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) + \frac{q_e}{m_e} (\boldsymbol{J} \times \boldsymbol{B}) - \frac{n_e q_e^2}{m_e} \hat{\eta} \cdot \boldsymbol{J} \tag{4.3}$$

- Where η is a tensor. This is *generalized Ohm's law*

Generalized Ohm's Law

- For a steady current in a uniform $\boldsymbol{E}$, $\partial \boldsymbol{J}/\partial t = 0,\ \nabla \cdot \boldsymbol{P} = 0$ and $\boldsymbol{B} = 0$ so that

$$\boldsymbol{E} = \eta \boldsymbol{J} \rightarrow \boldsymbol{J} = 1/\eta \boldsymbol{E}$$

- In general form, the electric field $\boldsymbol{E}$ can be found:

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B} - \frac{\boldsymbol{J} \times \boldsymbol{B}}{n_e q_e} + \frac{\nabla \cdot \boldsymbol{P}}{n_e q_e} + \hat{\eta} \cdot \boldsymbol{J} + \frac{m_e}{n_e q_e} \frac{\partial \boldsymbol{J}}{\partial t}$$

- Consider right hand side of this equation:
  - First term: $\boldsymbol{E}$ associated with plasma motion
  - Second term: Hall effect
  - Third term: Ambipolar diffusion from E-field generated by pressure gradients
  - Fourth term: Ohmic losses/Joule heating by resistivity
  - Fifth term: Electron inertia

One Fluid MHD Ohm's Law

- Generalized Ohm's law

$$\frac{\partial \boldsymbol{J}}{\partial t} = -\frac{q_e}{m_e}\nabla \cdot \boldsymbol{P}_e + \frac{n_e q_e^2}{m_e}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) + \frac{q_e}{m_e}(\boldsymbol{J} \times \boldsymbol{B}) - \frac{n_e q_e^2}{m_e}\hat{\eta} \cdot \boldsymbol{J}$$

- Now assume plasma is isotropic, so that $\nabla \cdot \boldsymbol{P} = \nabla p$
  Also we neglect Hall effect and Ambipolar diffusion in generalized Ohm's law since not important in one-fluid MHD.
  For slow variations, ***J*** = constant, so can write generalized Ohm's law as:

- Rearranging gives, $0 = \frac{n_e q_e^2}{m_e}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) - \frac{n_e q_e^2}{m_e}\eta \boldsymbol{J}$

$$\boxed{\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})}$$ One-fluid MHD Ohm's law

- Where $\sigma = 1/\eta$ is electrical conductivity

Simplified MHD Equations

- A set of simplified MHD equations can be written:

$$\begin{aligned}\frac{\partial \rho_m}{\partial t} + \nabla(\rho_m \boldsymbol{v}) &= 0 \\ \rho_m \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] &= -\nabla p + \boldsymbol{J}\times\boldsymbol{B} \\ \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} &= \eta \boldsymbol{J}\end{aligned}$$

- Fluid equations must be solved with reduced Maxwell equations

$$\nabla\times\boldsymbol{B} = \mu_0 \boldsymbol{J}, \quad \nabla\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$
$$\nabla\cdot\boldsymbol{B} = 0, \quad \nabla\cdot\boldsymbol{E} = 0$$

(displacement current term is ignored for low frequency phenomena)

- Here we have assumed that there is no accumulation of charge (i.e., $\rho_e = 0$)
- Complete set of equations only when *equation of state* for relationship between p and n (ρ) is specified.

$$p\rho_m^{-\gamma} = const$$

The Induction Equation

- Taking the curl of one-fluid MHD Ohm's law:

$$\nabla \times \boldsymbol{E} = -\nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \frac{1}{\sigma} \nabla \times \boldsymbol{J}$$

- Assuming σ=*const*. Substituting for $\boldsymbol{J} = \nabla \times \boldsymbol{B}/\mu_0$ from Ampere's law and using the law of induction equations (Faraday's law):

$$-\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \frac{1}{\mu_0 \sigma} \nabla \times (\nabla \times \boldsymbol{B})$$

- The double curl can be expanding from vector identity

$$-\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \frac{1}{\mu_0 \sigma} \nabla (\nabla \cdot \boldsymbol{B}) - \frac{1}{\mu_0 \sigma} \nabla^2 \boldsymbol{B}$$

- The second term in R.H.S. is zero by Gauss's law ($\nabla \cdot \boldsymbol{B} = 0$). So

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \boldsymbol{B}$$

MHD induction equation

The Induction Equation

- The MHD induction equation, together with fluid mass, momentum, and energy equations (EoS), a close set of equations for MHD state variables (ρ_m, $\boldsymbol{v}$, p, $\boldsymbol{B}$)

$$
\begin{aligned}
\frac{\partial \rho_m}{\partial t} + \nabla(\rho_m \boldsymbol{v}) &= 0 \\
\rho_m \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right] &= \frac{1}{\mu_0}(\boldsymbol{B} \cdot \nabla)\boldsymbol{B} - \nabla(p + \frac{B^2}{2\mu_0}) \\
p\rho_m^\gamma &= const \\
\frac{\partial \boldsymbol{B}}{\partial t} &= \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \boldsymbol{B}
\end{aligned}
$$

Here,

$$\boldsymbol{J} = \nabla \times \boldsymbol{B}/\mu_0$$

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B} + \boldsymbol{J}/\sigma$$

Ideal MHD

- In the case where the conductivity is very high ($\sigma \rightarrow \infty$), the electric field is ***E***= -***v*** x ***B*** (motional electric field only). It is known as *ideal Magnetohydrodynamics*.
- A set of equations:

$$\begin{aligned} \frac{\partial \rho_m}{\partial t} + \nabla(\rho_m \boldsymbol{v}) &= 0 \\ \rho_m \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right] &= \frac{1}{\mu_0}(\boldsymbol{B} \cdot \nabla)\boldsymbol{B} - \nabla(p + \frac{B^2}{2\mu_0}) \\ p\rho_m^\gamma &= const \\ \frac{\partial \boldsymbol{B}}{\partial t} &= \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \end{aligned}$$

- This is the most simplest assumption for MHD. But this is commonly used in Astrophysics.

Magnetic Field Behavior in MHD

- MHD induction equation: $$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \boldsymbol{B}$$
- $\nabla \times (\boldsymbol{v} \times \boldsymbol{B})$ Dominant: convection
  - Infinite conductivity limit: ideal MHD.
  - Flow and field are intimately connected. Field lines convect with the flow. (*flux fleezing*)
  - The flow response to the field motion via ***J*** x ***B*** force
- $(1/\mu_0 \sigma) \nabla^2 \boldsymbol{B}$ Dominant: Diffusion
  - Induction equation takes the form of a diffusion equation.
  - Field lines diffuse through the plasma down any field gradient
  - No coupling between magnetic field and fluid flow
  - Characteristic Diffusion time: $\tau = \mu_0 \sigma L^2 = \mu_0 L^2 / \eta$

Here using $\nabla = 1/L$

- Ratio of the convection term to the diffusion term:

$$R_m = \frac{\boldsymbol{v}\boldsymbol{B}/L}{\boldsymbol{B}/\mu_0 \sigma L^2} = \mu_0 \sigma \boldsymbol{v} L$$

Magnetic Reynold's number

Magnetic Field Behavior in MHD

- Rewrite continuity equation:

$$\frac{\partial \rho_m}{\partial t} = -\rho_m(\nabla \cdot \boldsymbol{v}) - (\boldsymbol{v} \cdot \nabla)\rho_m$$

  - first term describes compression (fluid contracts or expansion)
  - Second term describes advection
- The induction equation (ideal MHD) can be written as, using standard vector identities:

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{B}(\nabla \cdot \boldsymbol{v}) - (\boldsymbol{v} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{v}$$

- Equation is similar to continuity equation.
  - First term: compression
  - Second term: advection
  - Third term: new term describes stretching. It is related magnetic field amplification

Momentum Equation

- From equation of motion and continuity equations

$$\rho_m \frac{\partial \boldsymbol{v}}{\partial t} + \rho_m \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \frac{\partial}{\partial t}(\rho_m \boldsymbol{v}) + \boldsymbol{v} \nabla \cdot (\rho_m \boldsymbol{v}) + \rho_m \boldsymbol{v} \cdot \nabla \boldsymbol{v}$$
$$= \frac{\partial}{\partial t}(\rho_m \boldsymbol{v}) + \nabla \cdot (\rho_m \boldsymbol{v}\boldsymbol{v})$$

- Using definition of magnetic stress tensor, *the momentum equation* is

$$\frac{\partial}{\partial t}(\rho_m \boldsymbol{v}) + \nabla \cdot \left[\rho_m \boldsymbol{v}\boldsymbol{v} + \left(p + \frac{1}{2}B^2\right)\boldsymbol{I} - \boldsymbol{B}\boldsymbol{B}\right] = 0$$

$$\frac{\partial M}{\partial t} + \nabla \cdot \mathbf{\Pi} = 0$$

$\boldsymbol{I}$ is three-dimensional identity tensor

$$\mathcal{M}_i = \rho_m v_i$$ Momentum density

$$\Pi_{ij} = \rho_m v_i v_j + \left(p + \frac{1}{2}B^2\right)\delta_{ij} - B_i B_j = 0$$ Stress tensor

Conservation Form of Ideal MHD Eqs

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \boldsymbol{v}) = 0$$

Mass conservation

$$\frac{\partial}{\partial t}(\rho_m \boldsymbol{v}) + \nabla \cdot \left[\rho_m \boldsymbol{v}\boldsymbol{v} + \left(p + \frac{1}{2}B^2\right)\boldsymbol{I} - \boldsymbol{B}\boldsymbol{B}\right] = 0$$

Momentum conservation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_m v^2 + \rho_m e + \frac{1}{2}B^2\right)$$
$$+\nabla \cdot \left[\left(\frac{1}{2}\rho_m v^2 + \rho_m e + p + B^2\right)\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{B})\boldsymbol{B}\right] = 0$$

Energy conservation

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \cdot (\boldsymbol{v}\boldsymbol{B} - \boldsymbol{B}\boldsymbol{v}) = 0$$

induction equation

$$\nabla \cdot \boldsymbol{B} = 0$$

$$p = (\gamma - 1)\rho_m e$$

Ideal equation of state

Neglecting gravity force.

This form is often used in numerical simulation.

Poynting Flux

- From energy conservation equation, energy flux is

$$\boldsymbol{Y} \equiv \left(\frac{1}{2}\rho_m v^2 + \frac{\gamma}{\gamma - 1}p\right)\boldsymbol{v} + \frac{1}{\mu_0}(B^2\boldsymbol{v} - \boldsymbol{v}\cdot\boldsymbol{B}\boldsymbol{B})$$

- This compose hydrodynamic part and magnetic part.
- The magnetic part can be transformed:

$$\begin{aligned} \boldsymbol{Y}_{em} &\equiv \frac{1}{\mu_0}(B^2\boldsymbol{v} - \boldsymbol{v}\cdot\boldsymbol{B}\boldsymbol{B}) \\ &= -\frac{1}{\mu_0}(\boldsymbol{v}\times\boldsymbol{B})\times\boldsymbol{B} \\ &= \boxed{\boldsymbol{E}\times\boldsymbol{B}} \end{aligned}$$

- This is called *Poynting flux* (*Poynting vector*), which represents the flow of electromagnetic energy

Entropy conservation equation

- The best representation of the conservation form of MHD equation is in terms of the variables, ρ, $\boldsymbol{v}$, e and $\boldsymbol{B}$.
- A peculiar additional variable is the specific entropy s
- For adiabatic process of ideal gas, conservation of entropy is

$$\frac{DS}{Dt} \equiv \frac{\partial S}{\partial t} + (\boldsymbol{v} \cdot \nabla) S = 0$$

- But this is not in conservation form (but expresses the conservation of specific entropy co-moving with the fluid)
- A genuine conservation form is obtained by variable $\rho_m S$, the entropy per unit volume

$$\boxed{\frac{\partial}{\partial t}(\rho_m S) + \nabla \cdot (\rho_m S \boldsymbol{v}) = 0}$$

Entropy conservation equation

Hydro vs MHD

Newtonian MHD equation is shown the coupling of hydrodynamics with magnetic field

$$
\begin{aligned}
\frac{\partial \rho_m}{\partial t} + \nabla(\rho_m \boldsymbol{v}) &= 0 \\
\rho_m \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right] &= \frac{1}{\mu_0}(\boldsymbol{B} \cdot \nabla)\boldsymbol{B} - \nabla(p + \frac{B^2}{2\mu_0}) \\
p\rho_m^{\gamma} &= const \\
\frac{\partial \boldsymbol{B}}{\partial t} &= \nabla \times (\boldsymbol{v} \times \boldsymbol{B})
\end{aligned}
$$

MHD equation is recovered hydrodynamic equations when $B=0$.

$$
\begin{aligned}
\frac{\partial \rho_m}{\partial t} + \nabla(\rho_m \boldsymbol{v}) &= 0 \\
\rho_m \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right] &= -\nabla p \\
p\rho_m^{\gamma} &= const
\end{aligned}
$$

Hydro vs MHD

- Conservation form of Newtonian hydrodynamic equations

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \boldsymbol{v}) = 0$$

$$\frac{\partial}{\partial t}(\rho_m \boldsymbol{v}) + \nabla \cdot [\rho_m \boldsymbol{v}\boldsymbol{v} + p\boldsymbol{I}] = 0$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_m v^2 + \rho_m e\right) + \nabla \cdot \left[\left(\frac{1}{2}\rho_m v^2 + \rho_m e + p\right)\boldsymbol{v}\right] = 0$$

$$p = (\gamma - 1)\rho_m e$$

Summary

- Single fluid approach of plasma is called magnetohydrodynamics (MHD).
- In the case where the conductivity is very high, the electric field is $\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}$. It is known as ideal MHD.
- In ideal MHD, magnetic field is frozen into the fluid
- Lorentz force divides two different forces: magnetic pressure & curvature force
- The induction equation in ideal MHD shows evolution of magnetic field. It is including compression, advection and stretching
- The induction equation in resistive MHD includes diffusion of magnetic field.
- From energy conservation equation, energy flux composes hydrodynamic part and magnetic part. Magnetic part is called Poynting flux.