Lecture 2:

Numerical Methods for Solving MHD

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Conservation Form of Ideal MHD Eqs

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \boldsymbol{v}) = 0$$

Mass conservation

$$\frac{\partial}{\partial t}(\rho_m \boldsymbol{v}) + \nabla \cdot \left[\rho_m \boldsymbol{v}\boldsymbol{v} + \left(p + \frac{1}{2}B^2\right)\boldsymbol{I} - \boldsymbol{B}\boldsymbol{B}\right] = 0$$

Momentum conservation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_m v^2 + \rho_m e + \frac{1}{2}B^2\right)$$
$$+\nabla \cdot \left[\left(\frac{1}{2}\rho_m v^2 + \rho_m e + p + B^2\right)\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{B})\boldsymbol{B}\right] = 0$$

Energy conservation

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \cdot (\boldsymbol{v}\boldsymbol{B} - \boldsymbol{B}\boldsymbol{v}) = 0$$

induction equation

$$\nabla \cdot \boldsymbol{B} = 0$$

$$p = (\gamma - 1)\rho_m e$$

Ideal equation of state

Neglecting gravity force.

This form is often used in numerical simulation.

Conservation Form of Ideal MHD Eqs

hyperbolic system (Partial Differential Equations)

(with source terms)

$$\partial_t \boldsymbol{U} + \partial_j \boldsymbol{F}^i = 0$$ (without source terms)

$$\partial_t \boldsymbol{U} + \partial_j \boldsymbol{F}^i = \boldsymbol{S}$$

conserved variables

numerical flux

$$\boldsymbol{U} = \left[\begin{array}{c} \rho \\ \rho v^j \\ \frac{1}{2}\rho v^2 + \rho e + \frac{1}{2}B^2 \\ B^j \end{array} \right] ; \ \boldsymbol{F}^i = \left[\begin{array}{c} \rho v^i \\ \rho v^j v^i + (p + \frac{1}{2}B^2)\delta^i_j - B^j B^i \\ (\frac{1}{2}\rho v^2 + \rho e + p + B^2)v^i - (\boldsymbol{v}\cdot\boldsymbol{B})B^i \\ v^i B^j - v^j B^i \end{array} \right]$$

source term

contribution of gravity, radiation, resistivity etc.

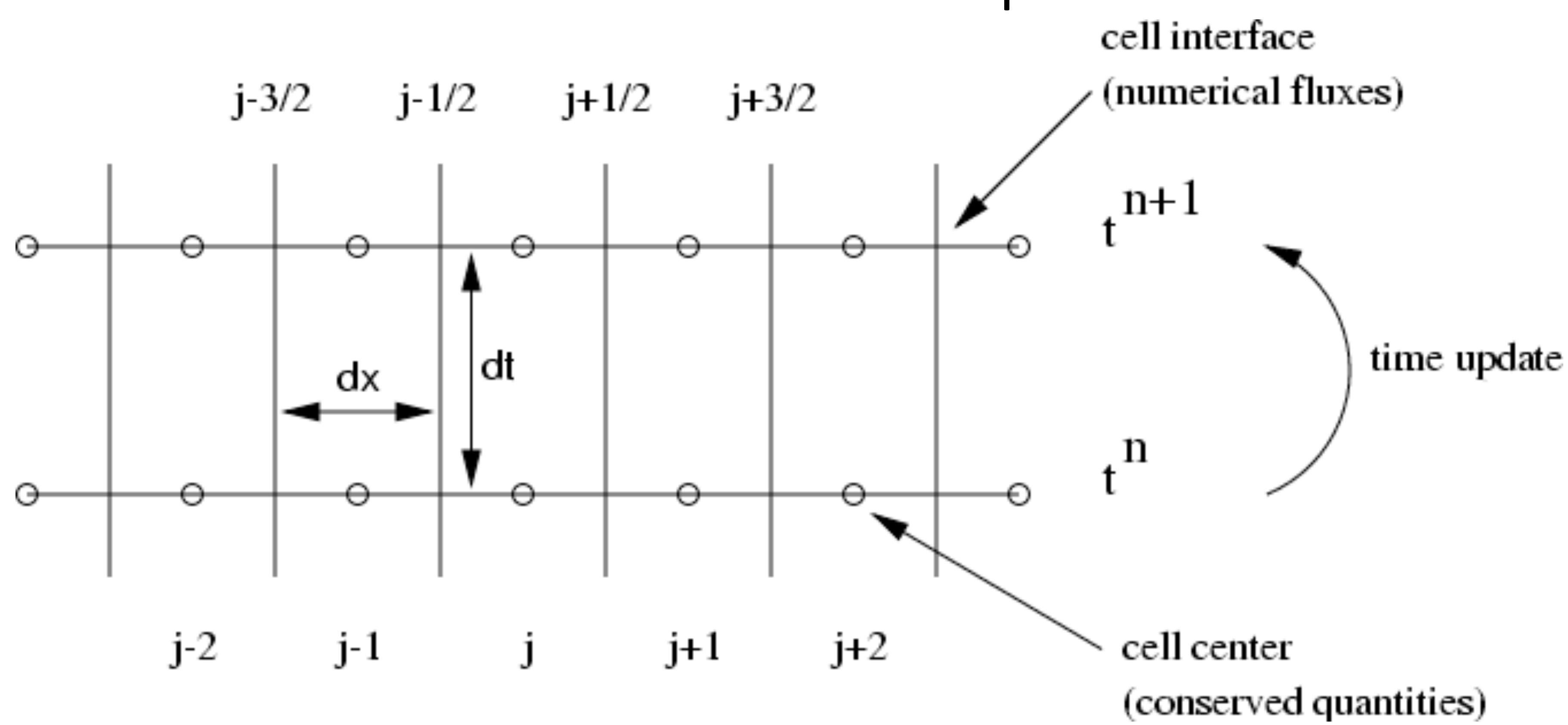
- Solving equations are 4 dimensions (time + 3 spatial directions).
- But we consider each spatial directions separately (Method of Lines).
- Here we mostly focus on 1 spatial dimension (t,x)

Finite Difference Scheme

- Partial differential equations (PDEs) are commonly solved numerically by approximating the derivatives with difference operators.
- Schemes of different orders can be obtained depending on the truncation of the corresponding Taylor series for the derivatives.
- Finite-difference schemes are based on a discretization of the x-t plane with a mesh of discrete points (t^n, x_j):

$$x_j = (j - 1/2)\Delta x, \quad t^n = n\Delta t, \quad j = 1, 2, \cdots \quad n = 0, 1, 2, \cdots$$

where Δx and Δt stand for the cell width and time step.



Finite Difference Scheme

- Let us consider the following scalar PDE

$$u_t + f_x = 0,\ u_0 = u(0,x),\ f = f(u)$$

- A Finite Difference scheme for this eq is a time-marching procedure to obtain approximations to the solution in the mesh points u_j^{n+1} from approximations in the previous time steps u_j^n
- We can approximate the time derivative with a first-order forward (Euler) difference and the spatial derivative with a first-order central difference

$$u_t = \frac{u_j^{n+1} - u_j^n}{\Delta t}$$

$$f_x = \frac{f_{j+1}^n - f_{j-1}^n}{2\Delta x}$$

which yields the explicit first-order central scheme:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}(f_{j+1}^n - f_{j-1}^n)$$

- Many other 1st-order and higher-order approximations are available in the literature.

Method of Lines

- Generic name of a family of discretization methods in which space and time variables are dealt with separately (3+1 approach).
- Time is discretized with finite differences and space discretization is done in various ways (finite differences, finite elements, spectral methods etc).
- The hydro equations can be written in a compact way as a "semi-discrete" system

$$\partial_t \boldsymbol{u} = \boldsymbol{S}$$

$\boldsymbol{u}$: array of dynamical fields

$\boldsymbol{S}$: remaining term in evolution eqs. including spatial derivatives

- PDE "disguised" as an ODE (standard ODE integrators can be applied) 1st-order forward in time (Euler step)

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^n + \Delta t \boldsymbol{S}(t^n, \boldsymbol{u}^n)$$

- 2nd-order Runge-Kutta scheme

$$\begin{aligned} \boldsymbol{u}^* &= \boldsymbol{u}^n + \Delta t \boldsymbol{S}^n \\ \boldsymbol{u}^{n+1} &= \frac{1}{2}\boldsymbol{u}^n + \frac{1}{2}\boldsymbol{u}^* + \frac{\Delta t}{2}\boldsymbol{S}^* \end{aligned}$$

- Higher-order in time ... many schemes available

Method of Lines

- Method of Lines can be used with any space discretization method. Finite-Difference scheme is a fairly standard choice.
- Field values at grid points are represented by the array

$$\boldsymbol{u}_{i,j,k} = \boldsymbol{u}(t, x_i, y_j, z_k)$$

- Space derivatives:

$$\begin{aligned}
2\partial_x u &\sim (u_{i+1,j,k} - u_{i-1,j,k})/\Delta x \\
2\partial_{xx} u &\sim (u_{i+1,j,k} + u_{i-1,j,k} - 2u_{i,j,k})/(\Delta x)^2 \\
2\partial_{xy} u &\sim (u_{i+1,j+1,k} - u_{i-1,j+1,k} \\
&\quad - u_{i+1,j-1,k} + u_{i-1,j-1,k})/(\Delta x \Delta y)
\end{aligned}$$

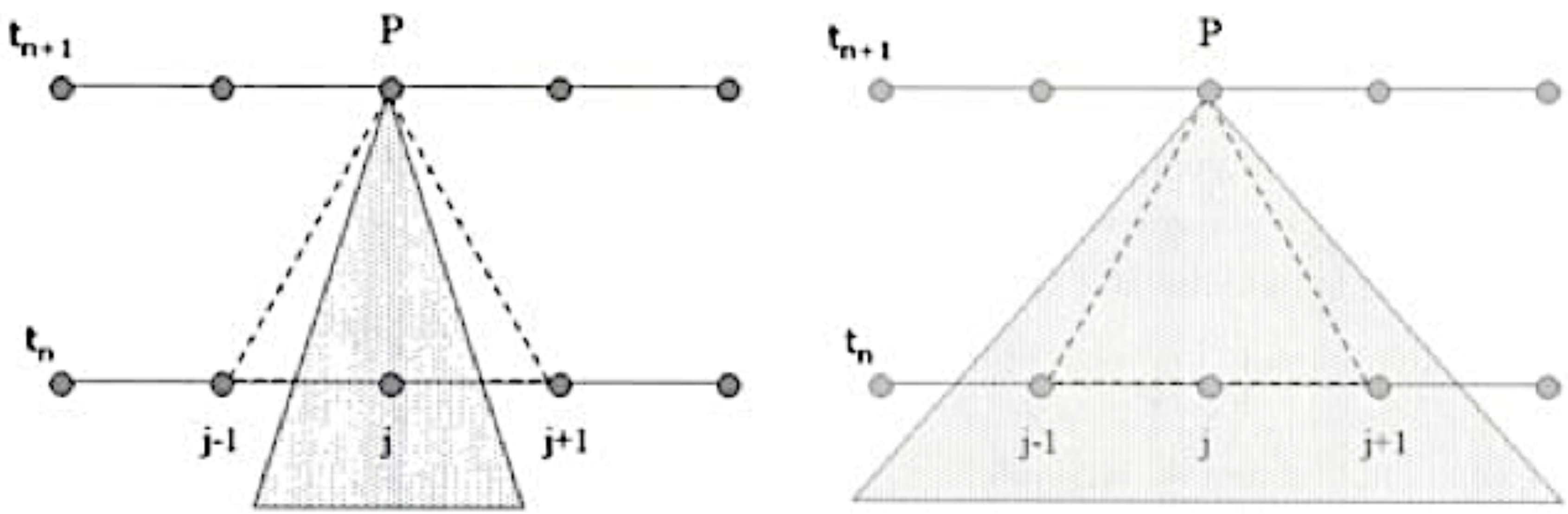
- Stencil: set of grid points needed to discretize space derivatives at a given point *P*.
  - Provides the numerical domain of dependence of selected point, i.e. any perturbation at one of the stencil points will change the computed value at *P* after a single time step.

Numerical propagation speed: $v^i_{\mathrm{num}} = s\dfrac{\Delta x^i}{\Delta t}$ s: stencil size

Method of Lines

- Physically, for a system describing wave propagation with some characteristic speeds, the field values at *P* are causally determined by the values inside the past half-cone with vertex at *P*, whose slope is given by the inverse of the largest characteristic speed of the system.

- This provides the physical domain of dependence of *P*.



- Courant (necessary) condition for numerical stability: the physical domain of dependence of *P* must be included in the numerical domain of dependence.

$$v_{\rm max} < n_i v^i_{\rm num}$$

Provides an upper limit for the numerical time step.

Characteristics

- The hydro equations involve wave propagation (hyperbolic equations). Remember they can be written in 1st-order quasilinear form:

$$\partial_t \boldsymbol{U} + \boldsymbol{A}\partial_x \boldsymbol{U} = 0 \qquad \boldsymbol{A}(\boldsymbol{U}) \equiv \partial \boldsymbol{F}/\partial \boldsymbol{U}$$ Jacobian matrix

- If Jacobian matrix has constant coefficients (linear case), the solution procedure is simple. First we diagonalize the Jacobian matrix so that

$$\boldsymbol{\Lambda} = \boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{R} \qquad \boldsymbol{\Lambda} = \mathrm{drag}(\lambda_1, \lambda_2, ..., \lambda_N)$$

eigenvectors matrix

eigenvalues matrix

- If we define the characteristic variables $\boldsymbol{W} \equiv \boldsymbol{R}^{-1}\boldsymbol{U}$
- We can decouple the original systems of equations:

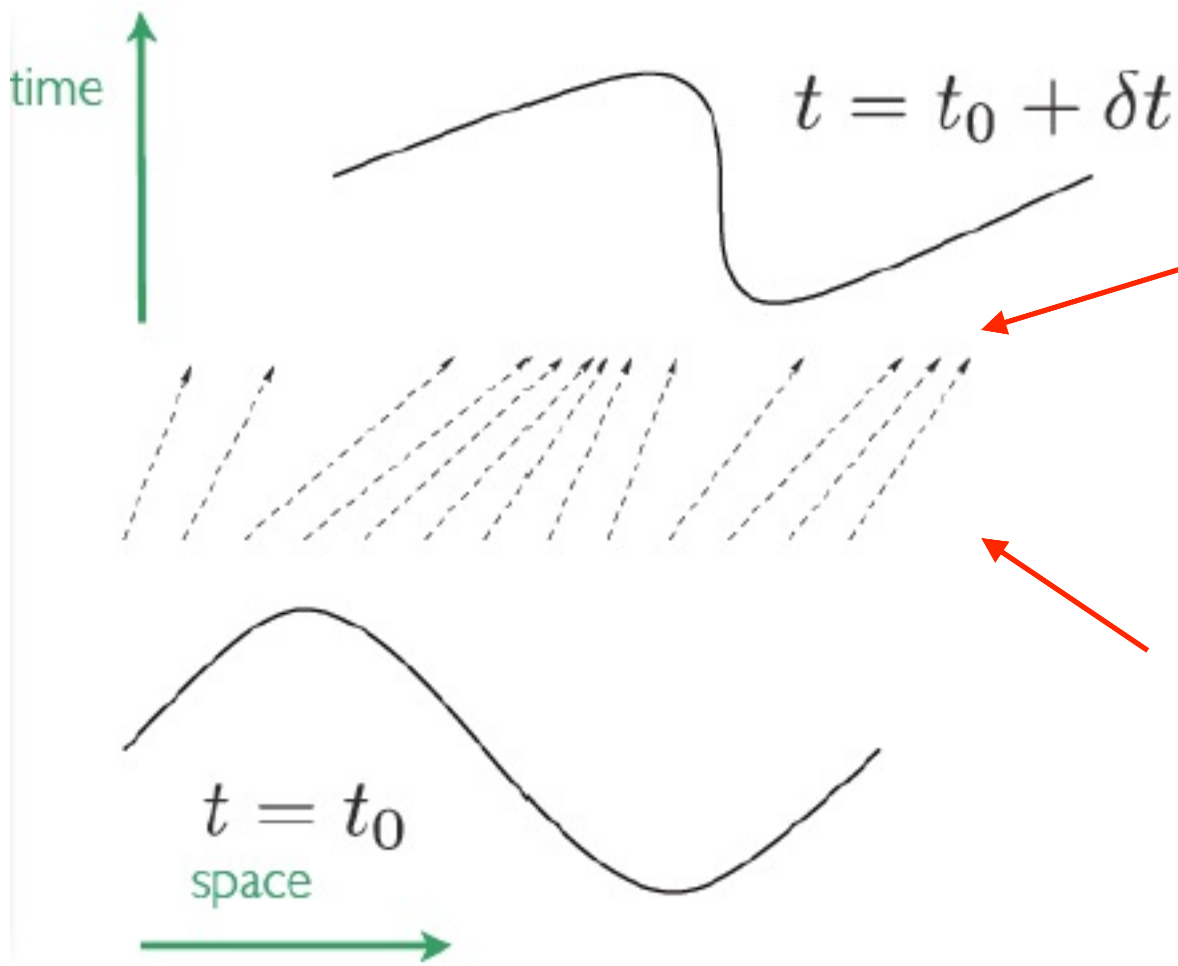
$$\partial_t \boldsymbol{W} + \boldsymbol{\Lambda}\partial_x \boldsymbol{W} = 0$$

$$\partial_t \bar{w}_i + \Lambda \partial_x \bar{w}_i = 0 \Longleftrightarrow \frac{d\bar{w}_i}{dt} = 0 \ \text{ along } \ \frac{\partial x}{\partial t} = \lambda_i(\boldsymbol{U}(x,t))$$

- Therefore, characteristic variables are constant along the curves of the (x,t) plane whose slopes are the corresponding eigenvalue.

Characteristics

- Such curves are called characteristic curves and their slopes are given locally by the characteristic speeds.



the characteristic speeds are different and the characteristic curves may "focus"

characteristic curves

- Since they are constant along the characteristics, the value of the characteristic variables at any given time is known once the initial value is known, that is

$$W^i(x,t) = W^i(x - \lambda_i t, t = 0)$$

Characteristics

- Once the solution is known in terms of the characteristic variables, it is straightforward to obtain the solution in terms of the original state vector:

$$\boldsymbol{W} = \boldsymbol{R}^{-1}\boldsymbol{U} \implies \boldsymbol{U} = \boldsymbol{R}\boldsymbol{W}$$

$$\boldsymbol{U}(x,t) = \sum_{i=1}^{N} W^i(x,t)R^{(i)} = \sum_{i=1}^{N} W^i(x-\lambda_i t, 0)R^(i)$$

- Thus, the solution is the linear superposition of N waves, each propagating independently of the rest with a speed given by the corresponding eigenvalue of the Jacobian matrix of the system.

- The so-called Godunov-type methods extend these concepts to nonlinear hyperbolic equations, solving Riemann problems of a new system of equations obtained by writing the original system as a quasi-linear system. Spectral information of Jacobian matrices is the basis of such solvers, as for linear systems.

Advection Equation

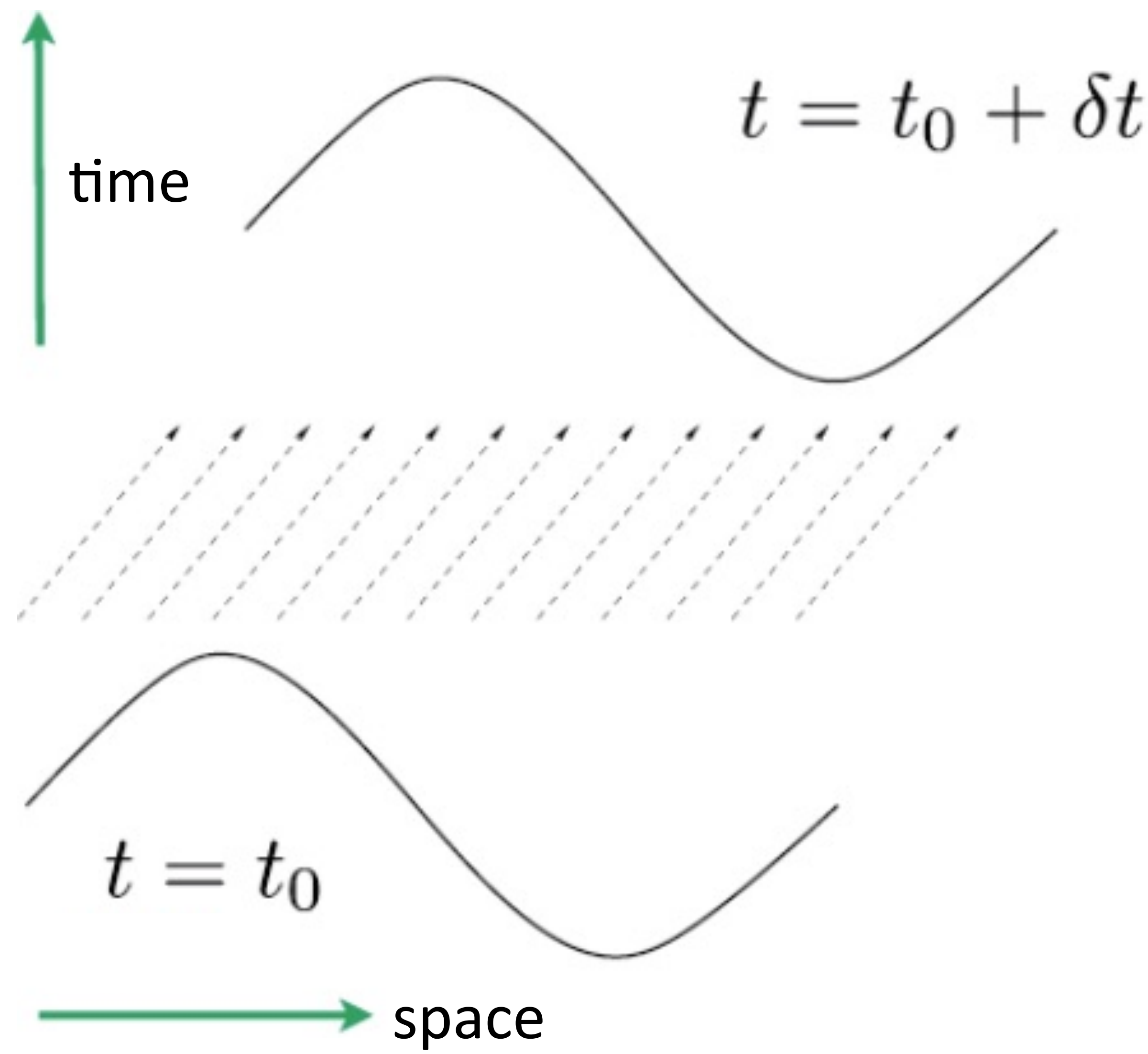
- Before discussing the solution of the hydrodynamics equations there are aspects of their nonlinear nature to point out.

- The simplest linear hyperbolic equation is the advection equation:

$$\partial_t u(x,t) + \partial_x u(x,t) = 0$$

- The solution is the initial one simply translated in space and time.

- The propagation speeds are constant in every point of space (linear nature of the equation).

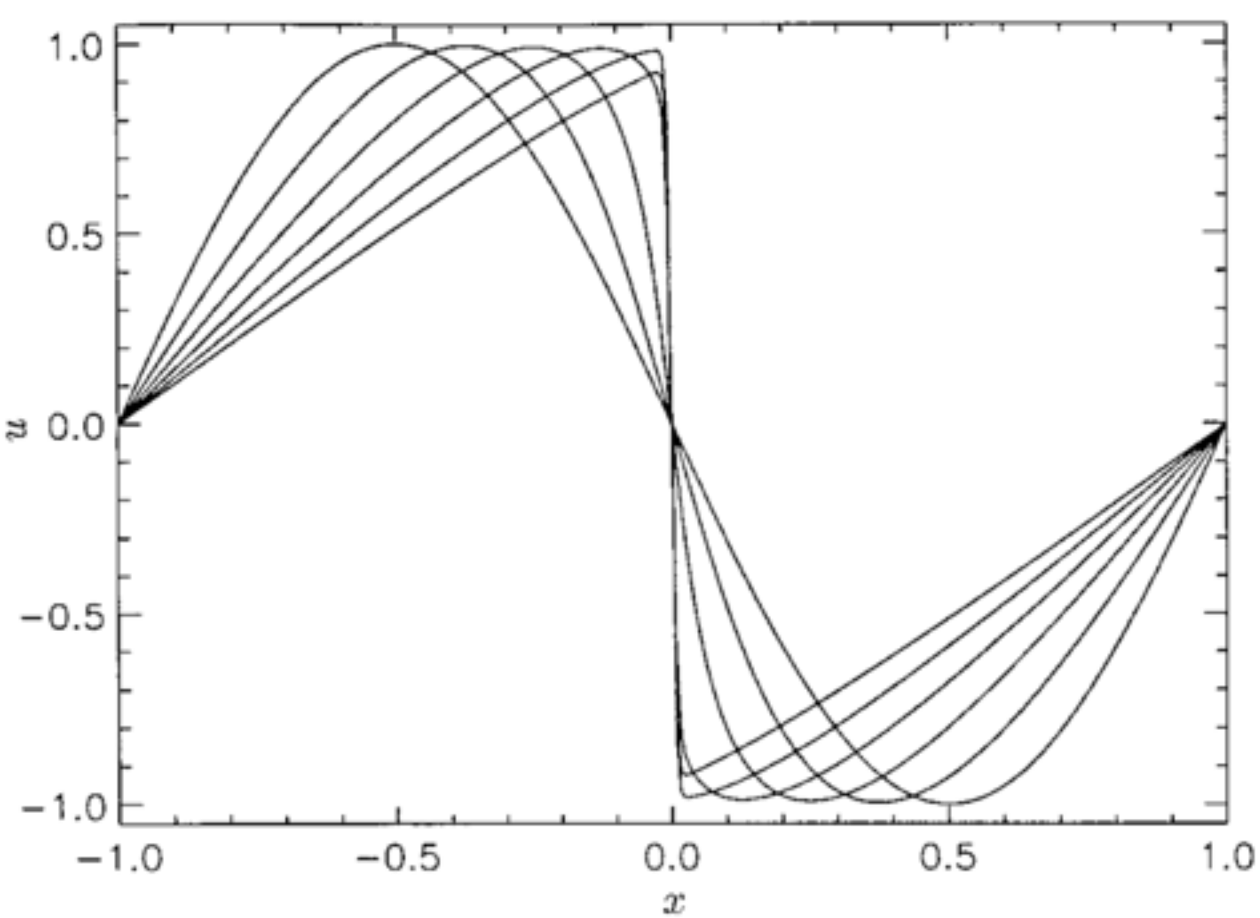


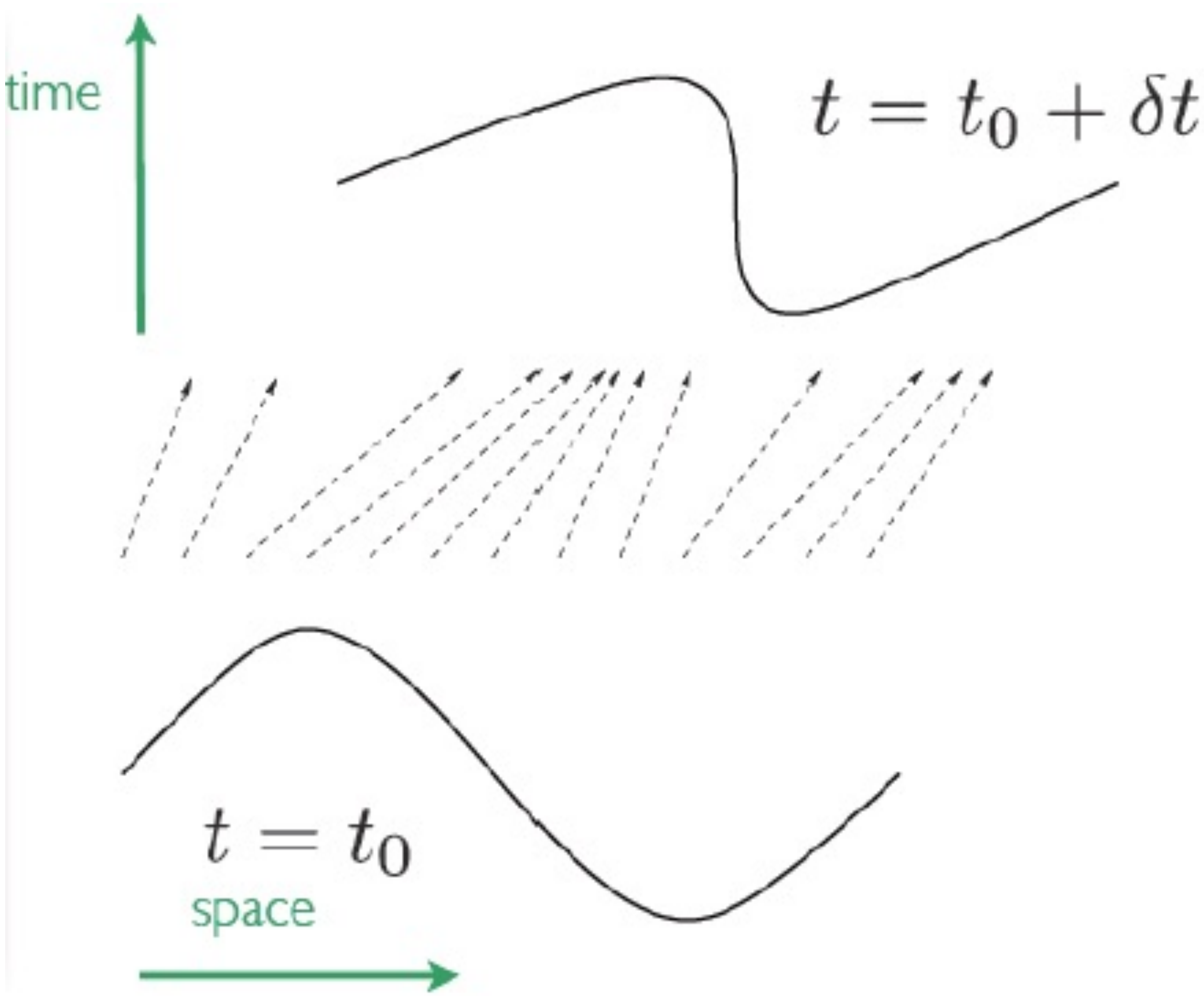
Burgers Equation

- The simplest nonlinear hyperbolic equation is Burgers equation:

$$\partial_t u(x,t) + u(x,t)\partial_x u(x,t) = \epsilon(x,t)\partial_x^2 u(x,t)$$

where the r.h.s. is zero in the inviscid limit. Despite the similarity with the advection equation, its solution is much different.





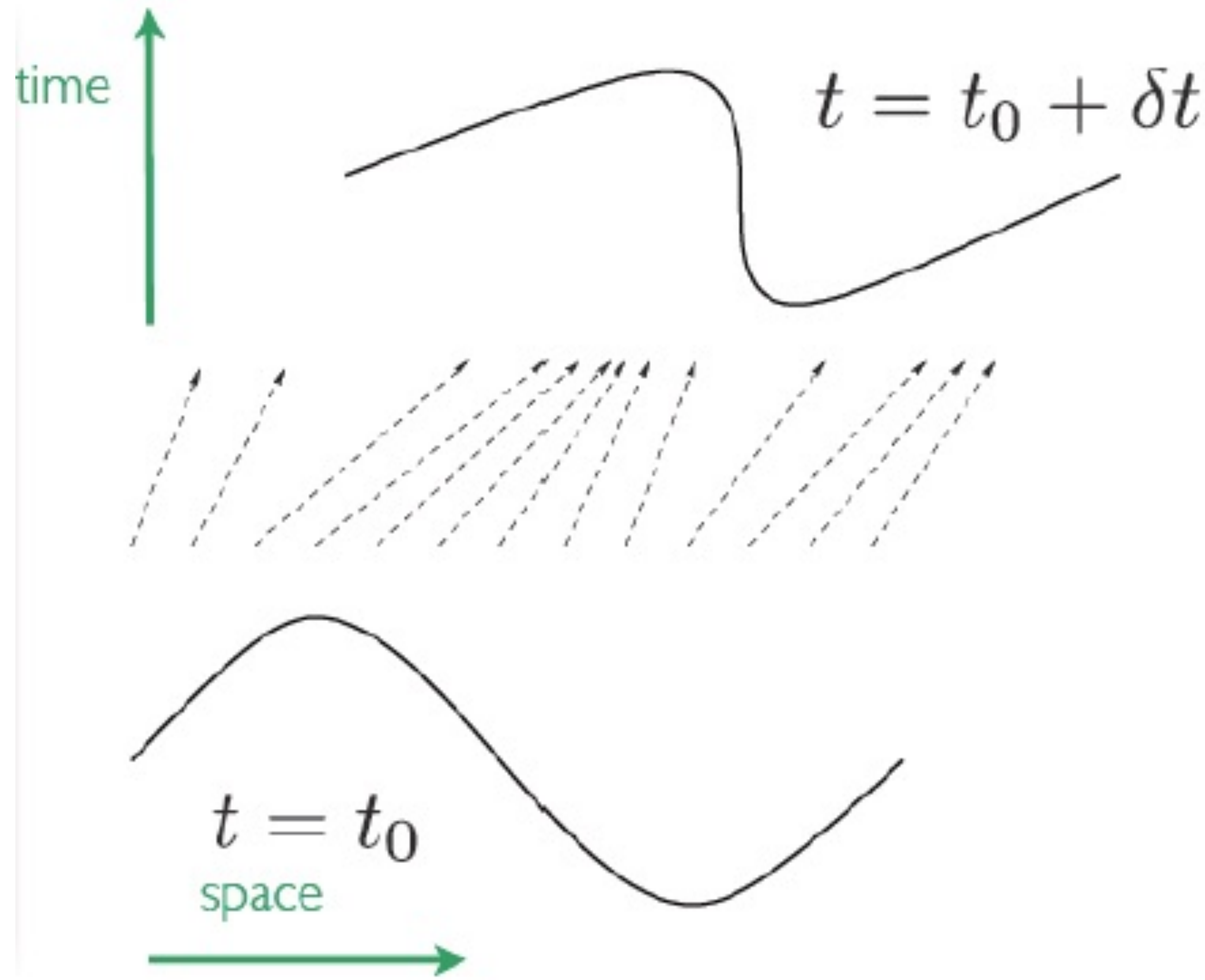
Burgers Equation

- This behaviour is known as "shock steepening" and is a consequence of the propagation speeds not being constant, contrary to what happens with the advection equation, but are functions of space and time (nonlinear nature of the equation).

- The maxima of the waves move faster than the minima and tend to reach them.

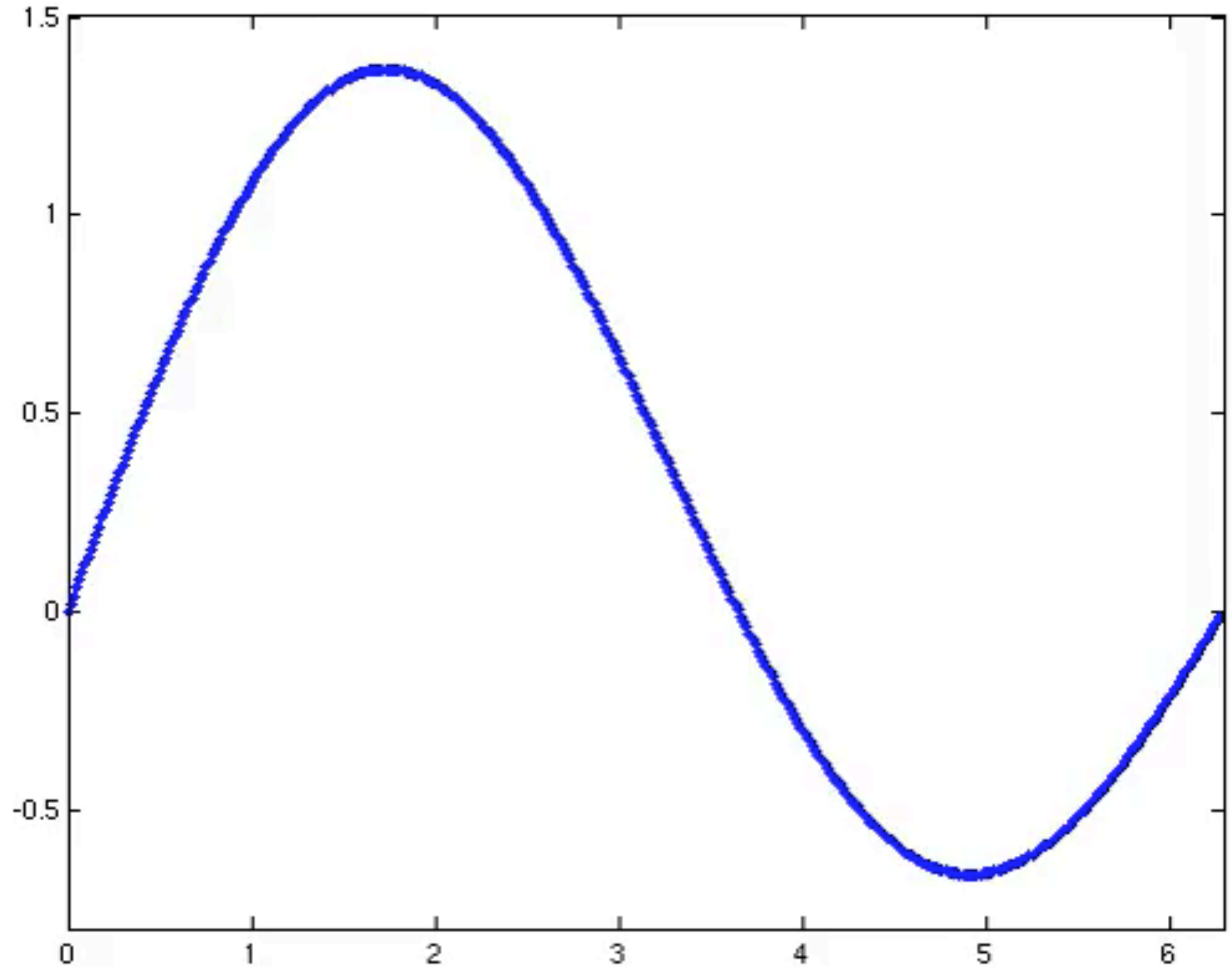
- NOTE: this is a property of the equations and not of the initial data.

- Even smooth initial data will lead to the appearance of shocks (in t>0) in the case of inviscid fluids.



Burgers Equation

$$\partial_t u + u\partial_x u = 0 \qquad u(x,0) = \sin x + \frac{1}{2}\sin\left(\frac{x}{2}\right)$$



Credit: Balbás & Tadmor.
CentPack (high-resolution central schemes)
http://www.cscamm.umd.edu/centpack

High-Resolution Methods

- High-resolution methods: modified high-order finite-difference methods with appropriate amount of numerical dissipation in the vicinity of a discontinuity.

- Quantity u^n_j is an approximation to $u(x_j, t^n)$ but in the case of a conservation law it is preferable to view it as an approximation to the average within the numerical cell $[x_{j-1/2}, x_{j+1/2}]$

$$\boldsymbol{u}_j^n \sim \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \boldsymbol{u}(x, t^n)dx$$

consistent with the integral form of the conservation law

- For hyperbolic systems of conservation laws, schemes written in conservation form guarantee that the convergence (if it exists) is to one of the so-called weak solutions of the original system of equations (Lax-Wendroff theorem 1960).

- A scheme written in conservation form reads:

$$\boldsymbol{u}_j^{n+1} = \boldsymbol{u}_j^n - \frac{\Delta t}{\Delta x}\left(\hat{\boldsymbol{f}}^n_{j+\frac{1}{2}} - \hat{\boldsymbol{f}}^n_{j-\frac{1}{2}}\right)$$

where $\hat{\boldsymbol{f}}^n_j$ is the numerical flux function

Conservative Form

- The conservation form of the scheme is ensured by starting with the integral version of the PDE in conservation form. By integrating the PDE within a spacetime computational cell $[x_{j-1/2}, x_{j+1/2}] \times [t^n, t^{n+1}]$
- the numerical flux function is an approximation to the time averaged flux across the interface:

$$\hat{\boldsymbol{f}}_{j+1/2} \sim \int_{t^n}^{t^{n+1}} \boldsymbol{f}(\boldsymbol{u}(x_{j+1/2}, t))dt$$

- The flux integral depends on the (unknown) solution at the numerical interfaces during the time step, $\boldsymbol{u}(x_{j+1/2}, t)$
- Key idea (Godunov 1959): a possible procedure is to calculate this solution by solving Riemann problems at every cell interface.

$$\boldsymbol{u}(x_{j+1/2}, t) = \boldsymbol{u}(0, \boldsymbol{u}_j^n, \boldsymbol{u}_{j+1}^n)$$

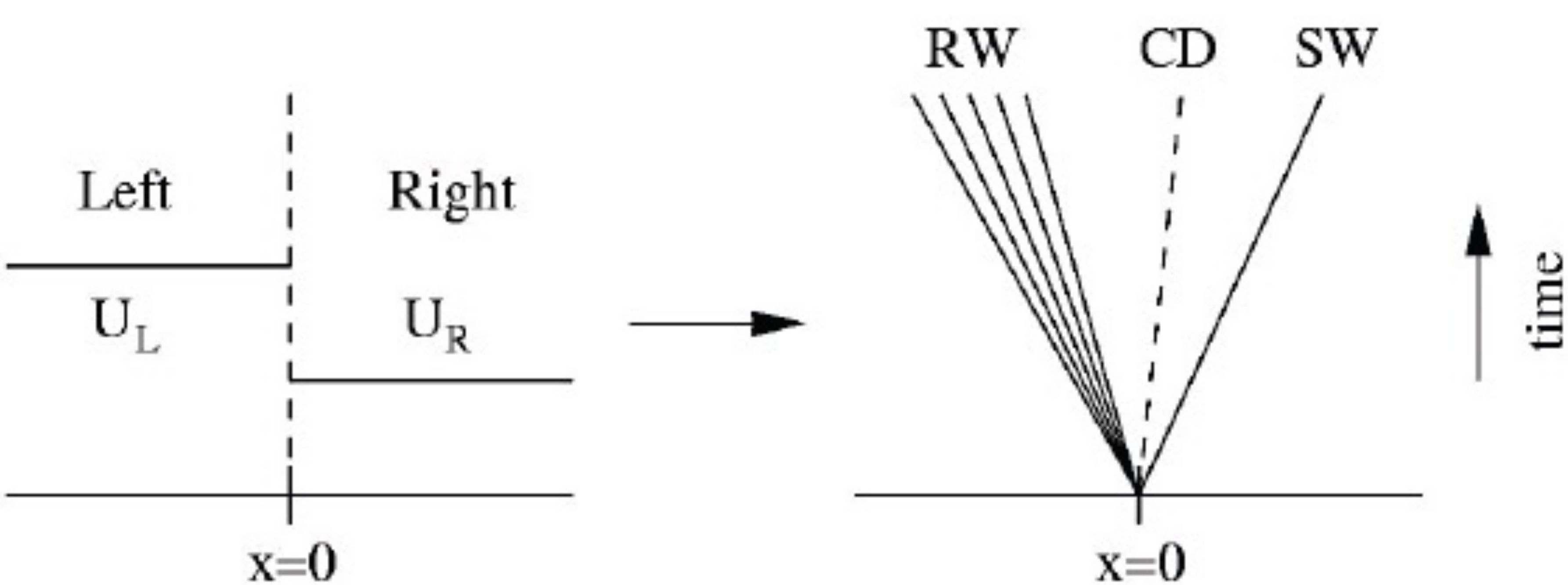
Riemann solution for the left and right states along the ray x/t=0.

The Riemann Problem

- A Riemann problem is an initial value problem with discontinuous initial data:

$$\boldsymbol{u}_0 = \begin{cases} \boldsymbol{u}_L & \text{if } x < 0 \\ \boldsymbol{u}_R & \text{if } x > 0 \end{cases}$$

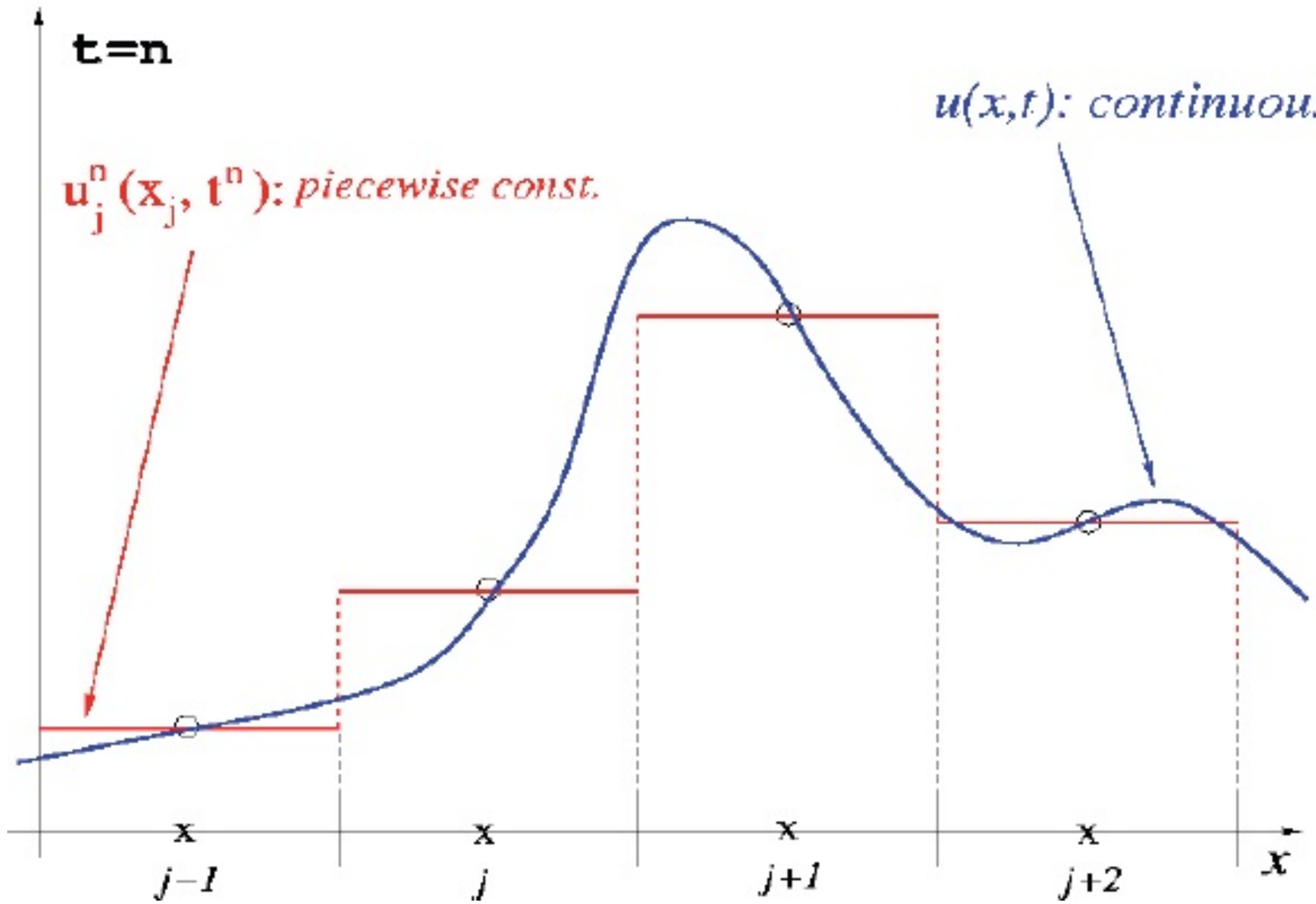
- The solution is constant along the straight lines $x/t = \text{constant}$, and, hence, self-similar.
- It consists of constant states separated by rarefaction waves (continuous self-similar solutions of the differential equations), shock waves, and contact discontinuities (Lax 1972).



The incorporation of the exact solution of Riemann problems to compute the numerical fluxes of Euler's equations is due to Godunov (1959)

The Riemann Problem

When a Cauchy problem described by a set of continuous PDEs is solved in a discretized form, the numerical solution is piecewise constant (collection of local Riemann problems).



The Riemann Problem

- This is particularly problematic when solving the hydrodynamic/MHD equations (either Newtonian or relativistic) for compressible fluids.
- Their hyperbolic, nonlinear character produces discontinuous solutions in a finite time (shock waves, contact discontinuities) even from smooth initial data!
- Any numerical scheme must be able to handle discontinuities in a satisfactory way.

Numerical Approach:

1. 1st-order accurate schemes (e.g., Lax-Friedrich): Non-oscillatory but inaccurate across discontinuities (excessive diffusion)

2. (standard) 2nd-order accurate schemes (e.g., Lax-Wendroff): Oscillatory across discontinuities

3. 2nd order accurate schemes with artificial viscosity

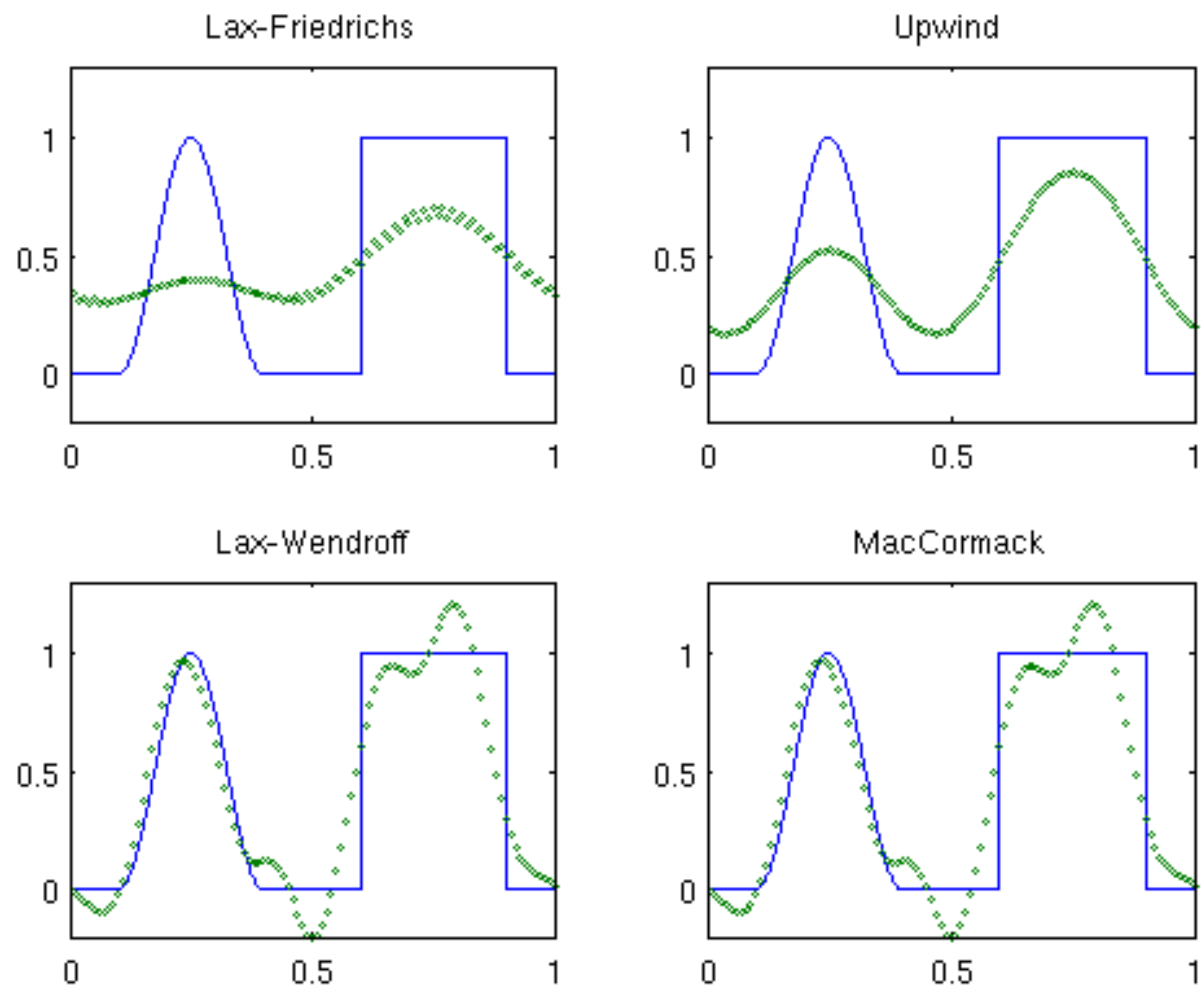
4. Godunov-type schemes (upwind High Resolution Shock Capturing schemes)

The Riemann Problem

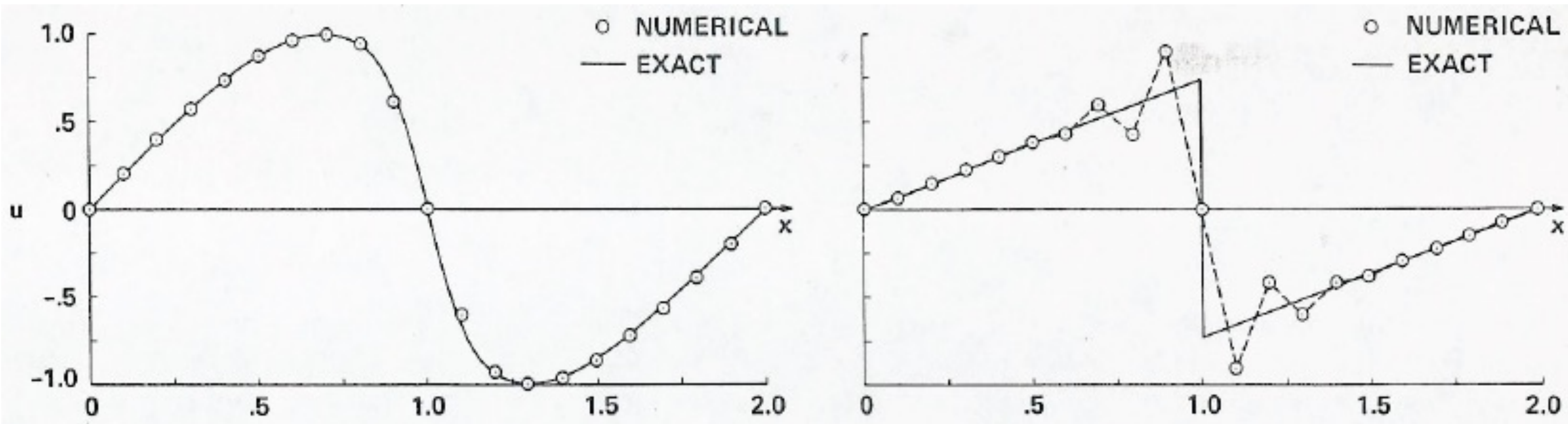
- Solving linear advection equations

1st-order:
smear out at even discontinuity

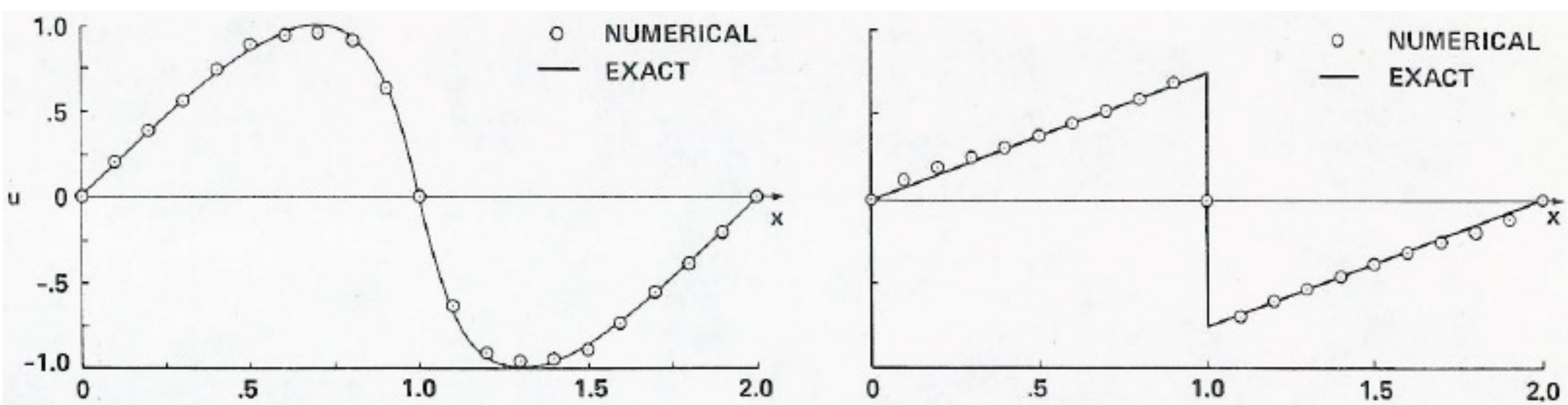
2nd-order:
oscillation at discontinuity



The Riemann Problem

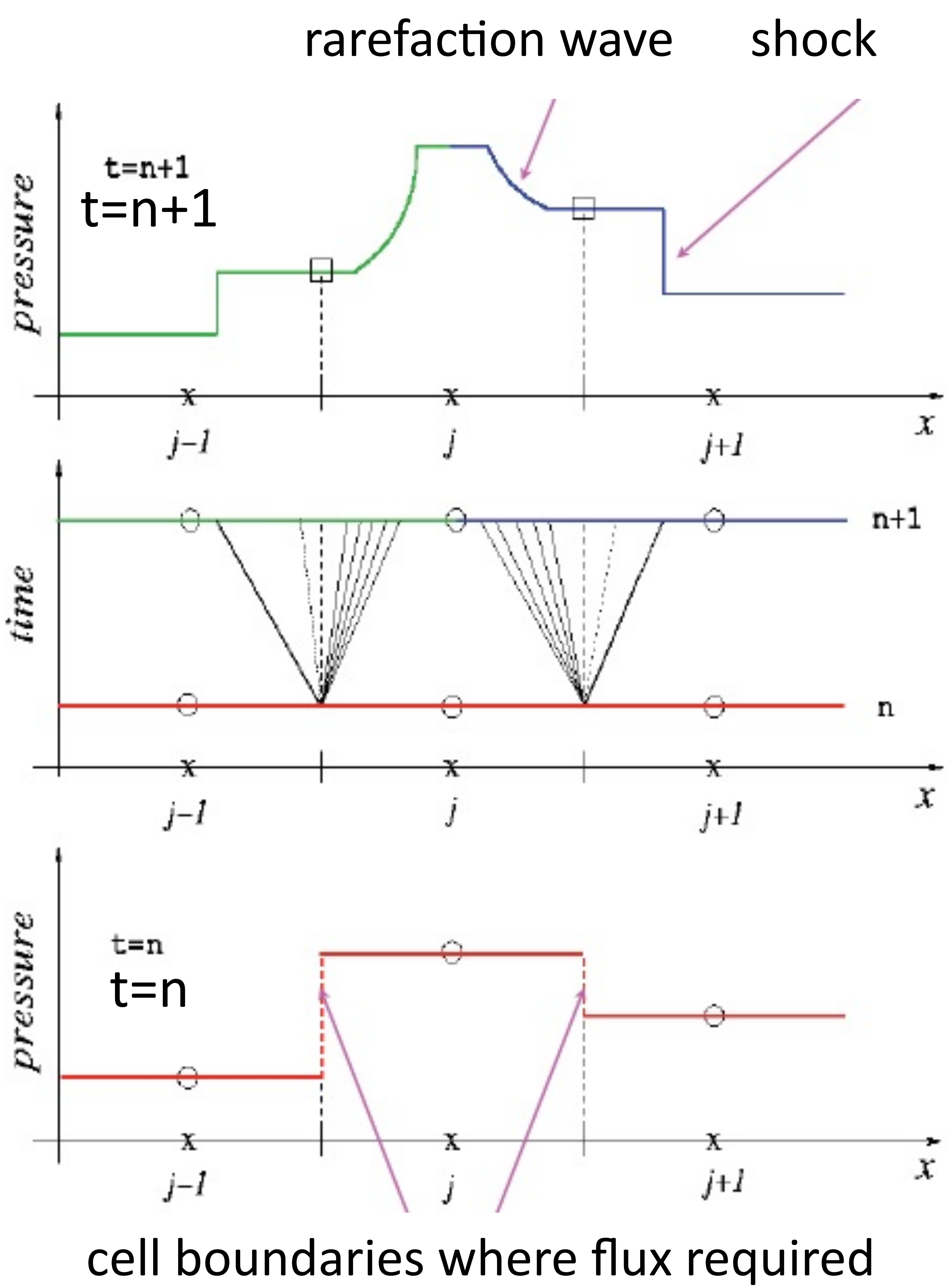


Lax-Wendroff numerical solution of Burger's equation at t=0.2 (left) and t=1.0 (right)



2nd order TVD numerical solution of Burger's equation at t=0.2 (left) and t=1.0 (right)

The Riemann Problem



Solution at time n+1 of the two Riemann problems at the cell boundaries $x_{j+1/2}$ and $x_{j-1/2}$

Spacetime evolution of the two Riemann problems at the cell boundaries $x_{j+1/2}$ and $x_{j-1/2}$. Each problem leads to a shock wave and a rarefaction wave moving in opposite directions

Initial data at time n for the two Riemann problems at the cell boundaries $x_{j+1/2}$ and $x_{j-1/2}$

$$\boldsymbol{u}_j^{n+1} = \boldsymbol{u}_j^n - \frac{\Delta t}{\Delta x}\left(\hat{\boldsymbol{f}}^n_{j+\frac{1}{2}} - \hat{\boldsymbol{f}}^n_{j-\frac{1}{2}}\right)$$

Approximate Riemann Solver

- In Godunov's method the structure of the Riemann solution is "lost" in the cell averaging process (1st order in space).

- The exact solution of a Riemann Problem is computationally expensive, particularly in multidimensions and for complicated EoS.

- This motivated development of approximate (linearized) Riemann solvers.

- Based on the exact solution of Riemann Problem corresponding to a new system of equations obtained by a linearization of the original one (quasilinear form). The spectral decomposition of the Jacobian matrices is on the basis of all solvers ("extending" ideas for linear systems).

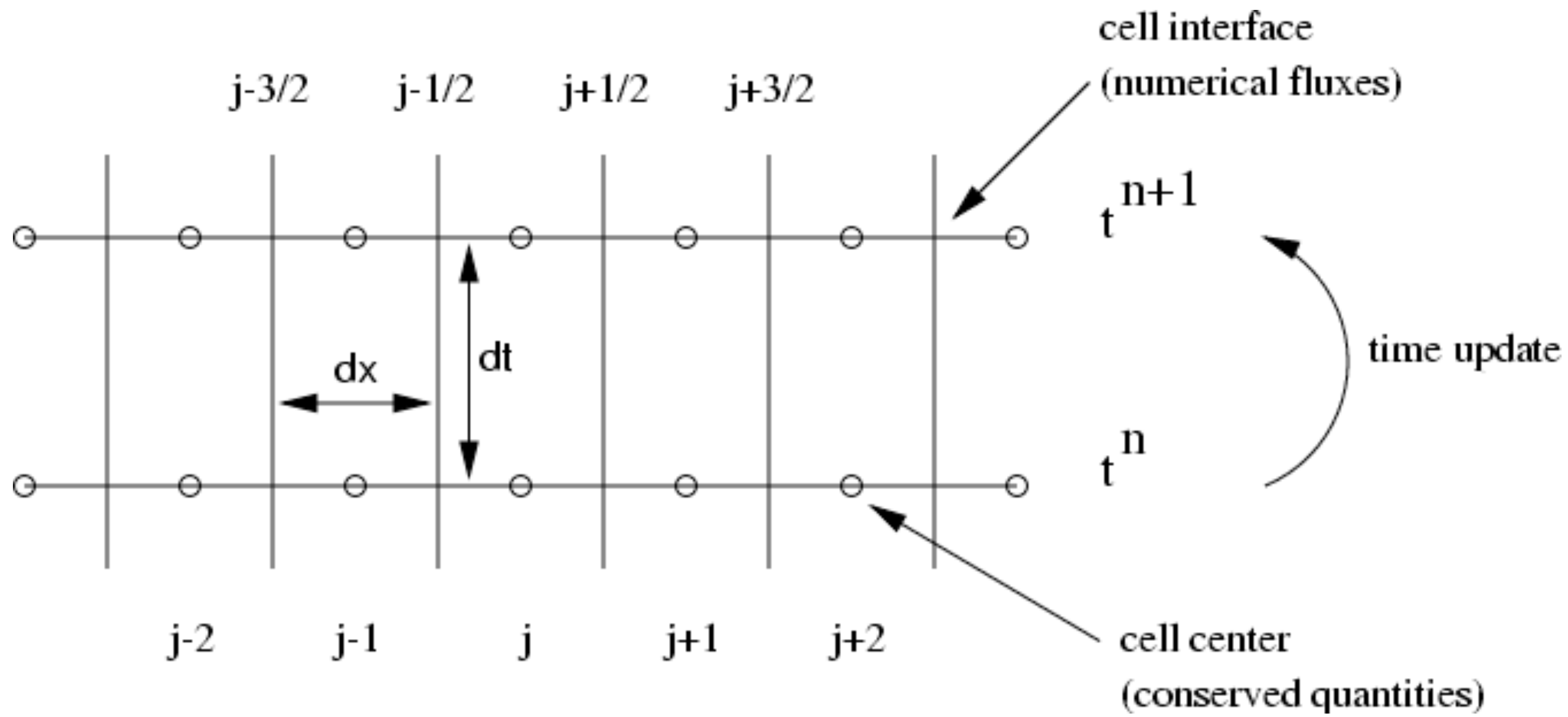
$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{f}}{\partial x} = 0 \Rightarrow \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{A}\frac{\partial \boldsymbol{u}}{\partial x} = 0, \;\; \boldsymbol{A} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}$$

- Approach followed by a subset of shock-capturing schemes, the so-called Godunov-type methods (Harten & Lax 1983; Einfeldt 1988).

Standard Implementation of a HRSC Scheme

1. Time update (Method of Line):

Algorithm in conserved form



$$\boldsymbol{u}_j^{n+1} = \boldsymbol{u}_j^n - \frac{\Delta t}{\Delta x}\left(\hat{\boldsymbol{f}}^n_{j+\frac{1}{2}} - \hat{\boldsymbol{f}}^n_{j-\frac{1}{2}}\right) + \Delta t \boldsymbol{S}_j^n$$

In practice: used 2nd or 3rd order time accurate, conservative Runge-Kutta schemes (Shu & Osher 1989; MoL)

Time Evolution

System of Conservation Equations

$$\partial_t \mathbf{U} = -\nabla \cdot \mathbf{F} + \mathbf{S} \equiv \mathbf{L}(\mathbf{U}).$$

We use multistep TVD Runge-Kutta method for time advance of conservation equations (RK2: 2nd-order, RK3: 3rd-order in time)

RK2, RK3: first step $\mathrm{U}^{(1)} = \mathrm{U}^n + \Delta t L(\mathrm{U}^n).$

RK2: second step (α=2, β=1)

$$\mathrm{U}^{n+1} = \frac{1}{\alpha}[\beta \mathrm{U}^n + \mathrm{U}^{(1)} + \Delta t L(\mathrm{U}^{(1)})],$$

RK3: second and third step (α=4, β=3)

$$\mathrm{U}^{(2)} = \frac{1}{\alpha}[\beta \mathrm{U}^n + \mathrm{U}^{(1)} + \Delta t L(\mathrm{U}^{(1)})],$$

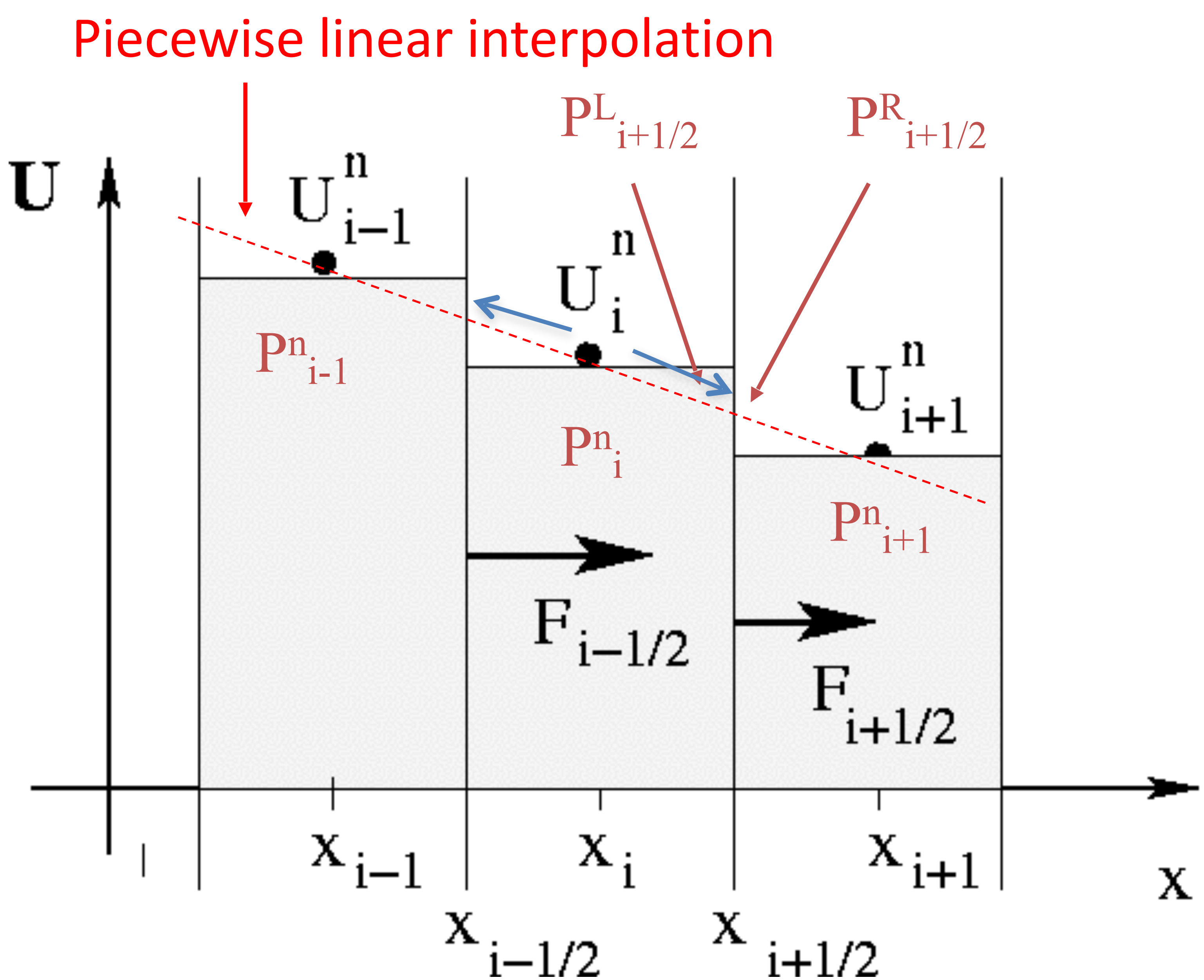
$$\mathrm{U}^{n+1} = \frac{1}{\beta}[\beta \mathrm{U}^n + 2\mathrm{U}^{(2)} + 2\Delta t L(\mathrm{U}^{(2)})],$$

Standard Implementation of a HRSC Scheme

2.cell reconstruction:

Cell-centered variables (P_j)
→ interpolate to right and left side of Cell-interface variables ($P^L_{j+1/2}$, $P^R_{j+1/2}$)

Piecewise constant (Godunov), linear (MUSCL, MC, van Leer), parabolic (PPM), or higher interpolation (WENO, MP) procedures are used.



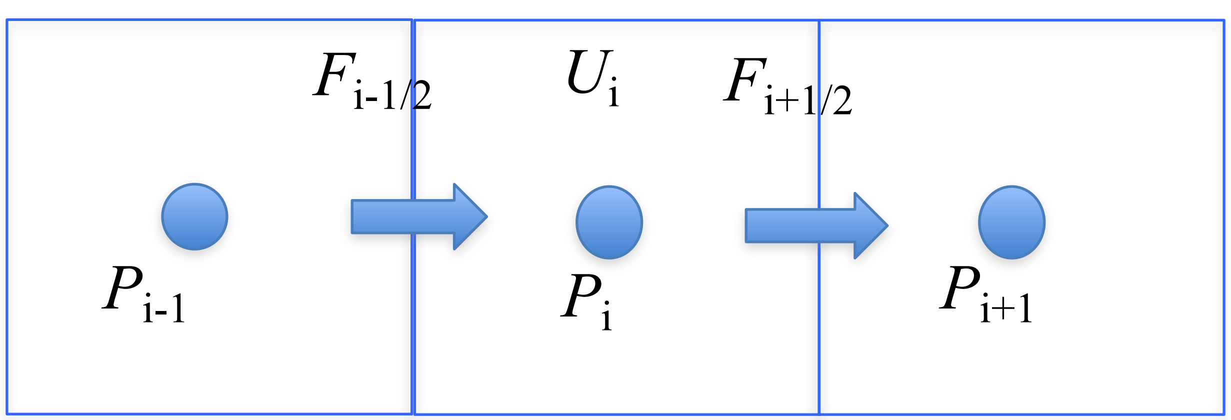
Standard Implementation of a HRSC Scheme

3.numerical fluxes:

• Calculate numerical flux at cell-inteface from reconstructed cell-interface variables based on Riemann problem

• Approximated Riemann Solvers (Roe, HLLE, HLLC, HLLD,…) are used

• Explicit use of spectral information of system (HLLE use only the maximum left- and right- going wave speeds)

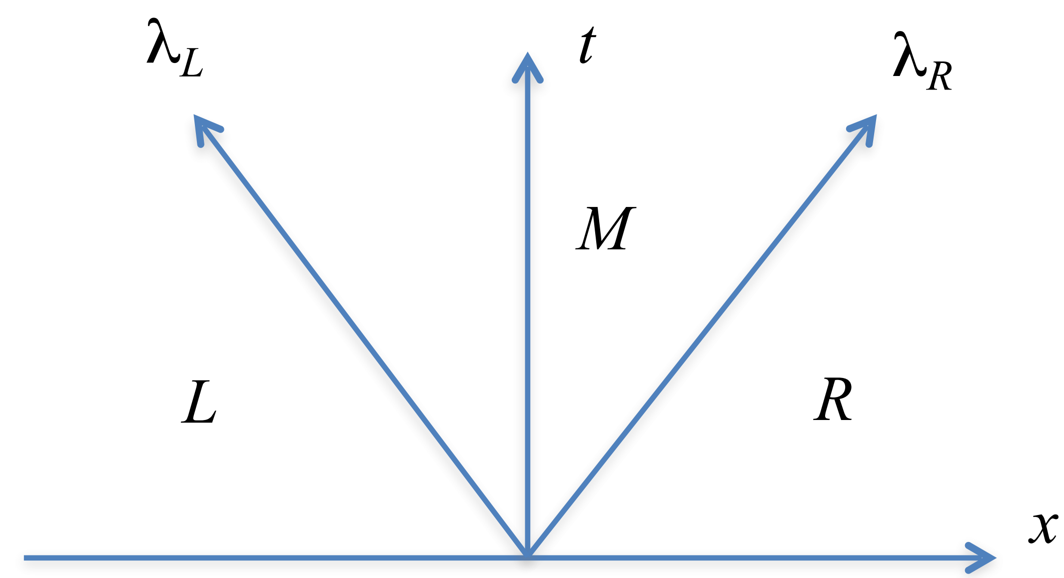


λ_R, λ_L: fastest characteristic speed

$$\begin{aligned} \lambda_R &= \max(\lambda_R, 0) \\ \lambda_L &= \min(\lambda_L, 0) \end{aligned}$$

HLL flux

$$\mathcal{F}^{HLL} = \frac{\lambda_R \mathcal{F}_L - \lambda_L \mathcal{F}_R + \lambda_R \lambda_L (\mathcal{U}_R - \mathcal{U}_L)}{\lambda_R - \lambda_L}$$



If $\lambda_L > 0$ $\qquad F_{HLL} = F_L$

$\lambda_L < 0 < \lambda_R$, $F_{HLL} = F_M$

$\lambda_R < 0$ $\qquad F_{HLL} = F_R$

Approximate Riemann Solver

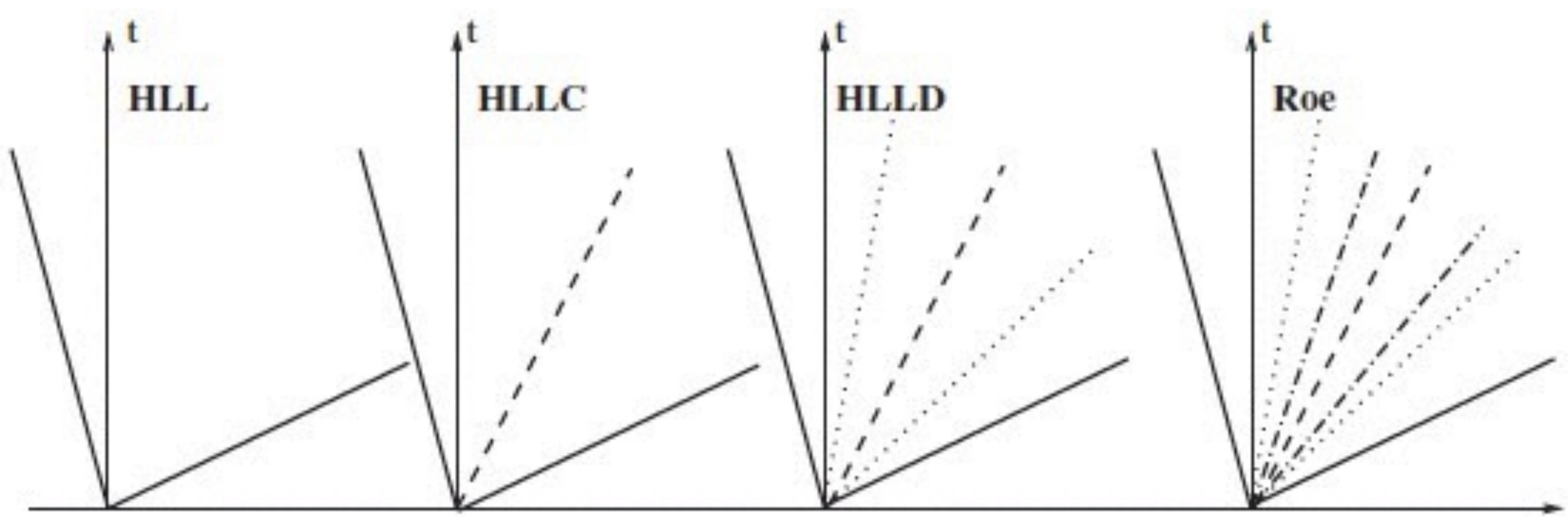
- (Lax-Friedrich Riemann solver: simpler version of HLL Riemann solver)

- HLLE Approximate Riemann solver: single state in Riemann fan

- HLLC Approximate Riemann solver: two-state in Riemann fan (Mignone & Bodo 2006, Honkkila & Janhunen 2007)

- HLLD Approximate Riemann solver: six-state in Riemann fan (Mignone et al. 2009)

- Roe-type full wave decomposition Riemann solver (Anton et al. 2010)



Constrained Transport

Differential Equations

$$\frac{1}{c}\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} = 0$$
$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\longrightarrow \quad \frac{\partial(\vec{\nabla} \cdot \vec{B})}{\partial t} = 0$$

- The evolution equation can keep divergence free magnetic field

• If treat the induction equation as all other conservation laws, it can not maintain divergence free magnetic field

=> We need spatial treatment for magnetic field evolution

Constrained transport scheme

• Evans & Hawley's Constrained Transport (need staggered mesh)
• Flux interpolated constrained transport (flux-CT) (Toth 2000)
• Fixed Flux-CT, Upwind Flux-CT (Gardiner & Stone 2005, 2007)
• and more higher ordered CT schemes

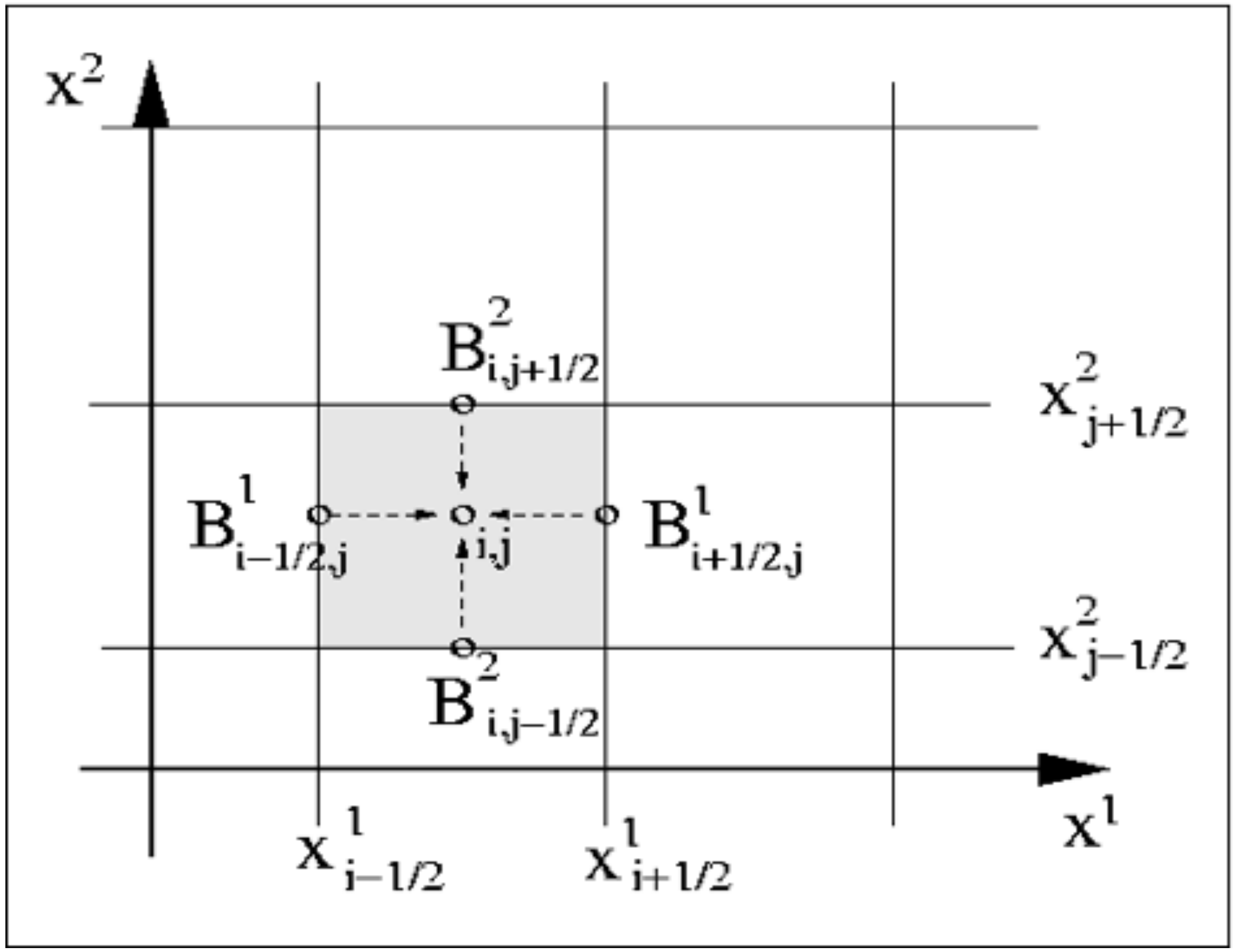
Other method

• Diffusive cleaning (GLM formulation)

Staggered Constrained Transport

Use staggered grid (with B defined at the cell-interfaces) and evolve magnetic fluxes through the cell interfaces using the electric field evaluated at the cell-edges.

This keeps the following "cell-centred" numerical representation of $\mathrm{div}B$ invariant

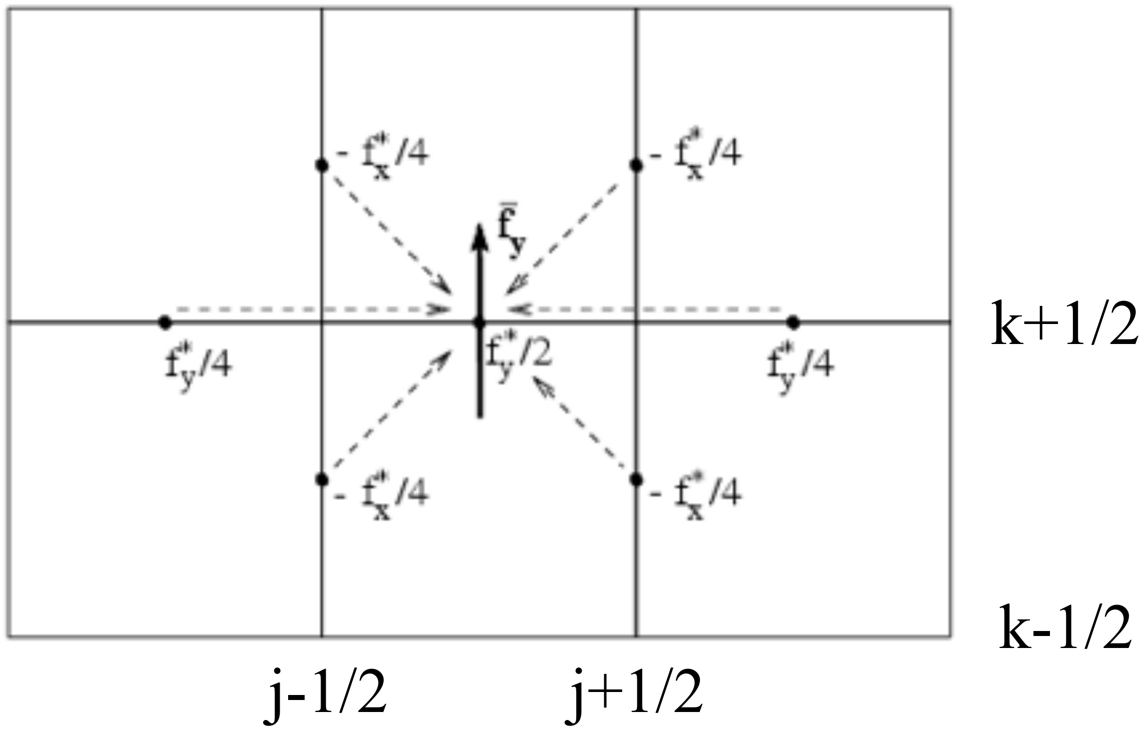


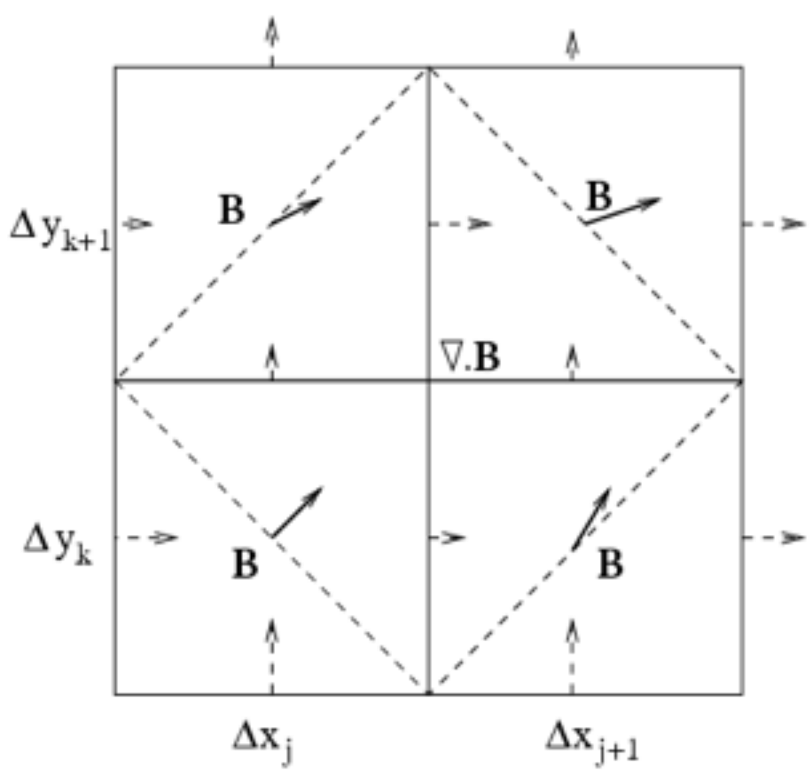
$$\vec{\nabla}\cdot\vec{B}_{i,j} = \frac{(B^1_{i+1/2,j} - B^1_{i-1/2,j})}{\Delta x^1} + \frac{(B^2_{i,j+1/2} - B^2_{i,j-1/2})}{\Delta x^2} :$$

Flux Interpolated Constrained Transport

2 D case

Toth (2000)





Use the "modified flux" $\bar{f}$ that is such a linear combination of normal fluxes at neighbouring interfaces that the "corner-centred" numerical representation of $\mathrm{div}B$ is kept invariant during integration.

$$B^{x,n+1}_{j,k} = B^{x,n}_{j,k} - \Delta t \frac{\bar{f}^y_{j,k+1/2} - \bar{f}^y_{j,k-1/2}}{\Delta y}$$

$$B^{y,n+1}_{j,k} = B^{y,n}_{j,k} - \Delta t \frac{\bar{f}^x_{j+1/2,k} - \bar{f}^x_{j-1/2,k}}{\Delta x}$$

$$\begin{aligned}\bar{f}^x_{j+1/2,k} = \frac{1}{8}\big(\ & 2\ f^{x,*}_{j+1/2,k} + f^{x,*}_{j+1/2,k+1} + f^{x,*}_{j+1/2,k-1} \\ & - f^{y,*}_{j,k+1/2} - f^{y,*}_{j+1,k+1/2} - f^{y,*}_{j,k-1/2} - f^{y,*}_{j+1,k-1/2}\big)\end{aligned}$$

$$\begin{aligned}\bar{f}^y_{j,k+1/2} = \frac{1}{8}\big(\ & 2\ f^{y,*}_{j,k+1/2} + f^{y,*}_{j+1,k+1/2} + f^{y,*}_{j-1,k+1/2} \\ & - f^{x,*}_{j+1/2,k} - f^{x,*}_{j+1/2,k+1} - f^{x,*}_{j-1/2,k} - f^{x,*}_{j-1/2,k+1}\big)\end{aligned}$$

$$(\nabla \cdot \boldsymbol{B})_{j,k} = \frac{B^x_{j+1,k} - B^x_{j-1,k}}{2\Delta x} + \frac{B^y_{j,k+1} - B^y_{j,k-1}}{2\Delta y}$$

General (Approximate) EoS

Mignone & McKinney (2007)

- In the theory of relativistic perfect single gases, specific enthalpy is a function of temperature alone (Synge 1957)

$$h = \frac{K_3(1/\Theta)}{K_2(1/\Theta)}\,,$$

Θ: temperature p/ρ
K_2, K_3: the order 2 and 3 of modified Bessel functions

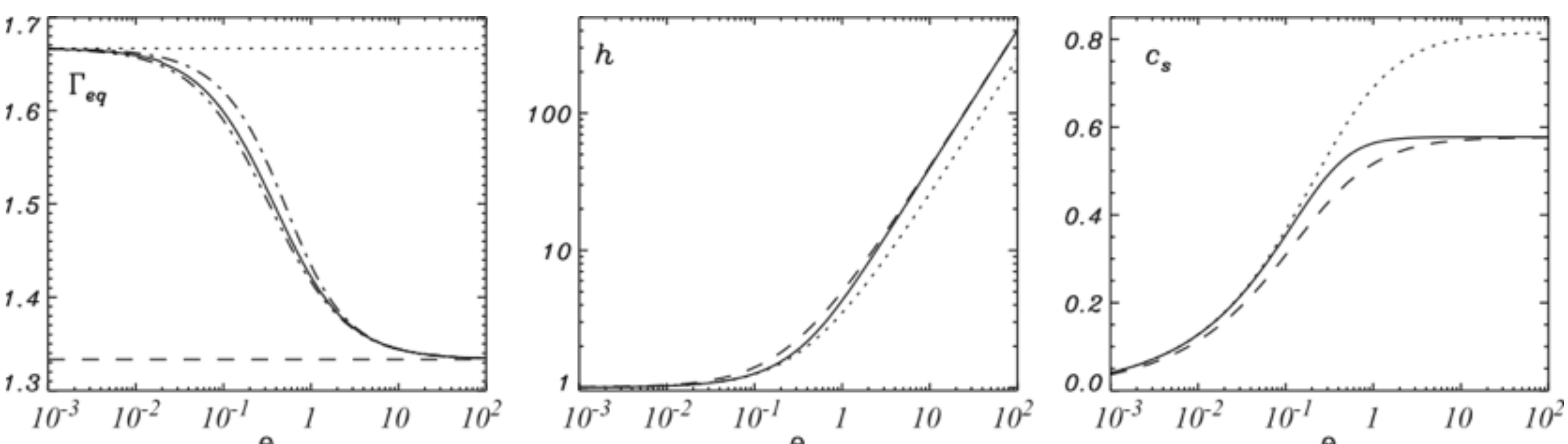
- Constant Γ-law EoS (ideal EoS) : $\boxed{h = 1 + \frac{\Gamma}{\Gamma - 1}\Theta}$
    - Γ: constant specific heat ratio
- Taub's fundamental inequality(Taub 1948) $(h-\Theta)\,(h-4\Theta) \geqslant 1\,,$

$$\Gamma_{\rm eq} = \frac{h-1}{h-1-\Theta}\,,$$

$\Theta \rightarrow 0$, $\Gamma_{\rm eq} \rightarrow 5/3$, $\Theta \rightarrow \infty$, $\Gamma_{\rm eq} \rightarrow 4/3$

- TM EoS (approximate Synge's EoS) (Mignone et al. 2005) $\boxed{h = \frac{5}{2}\Theta + \sqrt{\frac{9}{4}\Theta^2 + 1}\,,}$

Solid: Synge EoS
Dotted: ideal + Γ=5/3
Dashed: ideal+ Γ=4/3
Dash-dotted: TM EoS



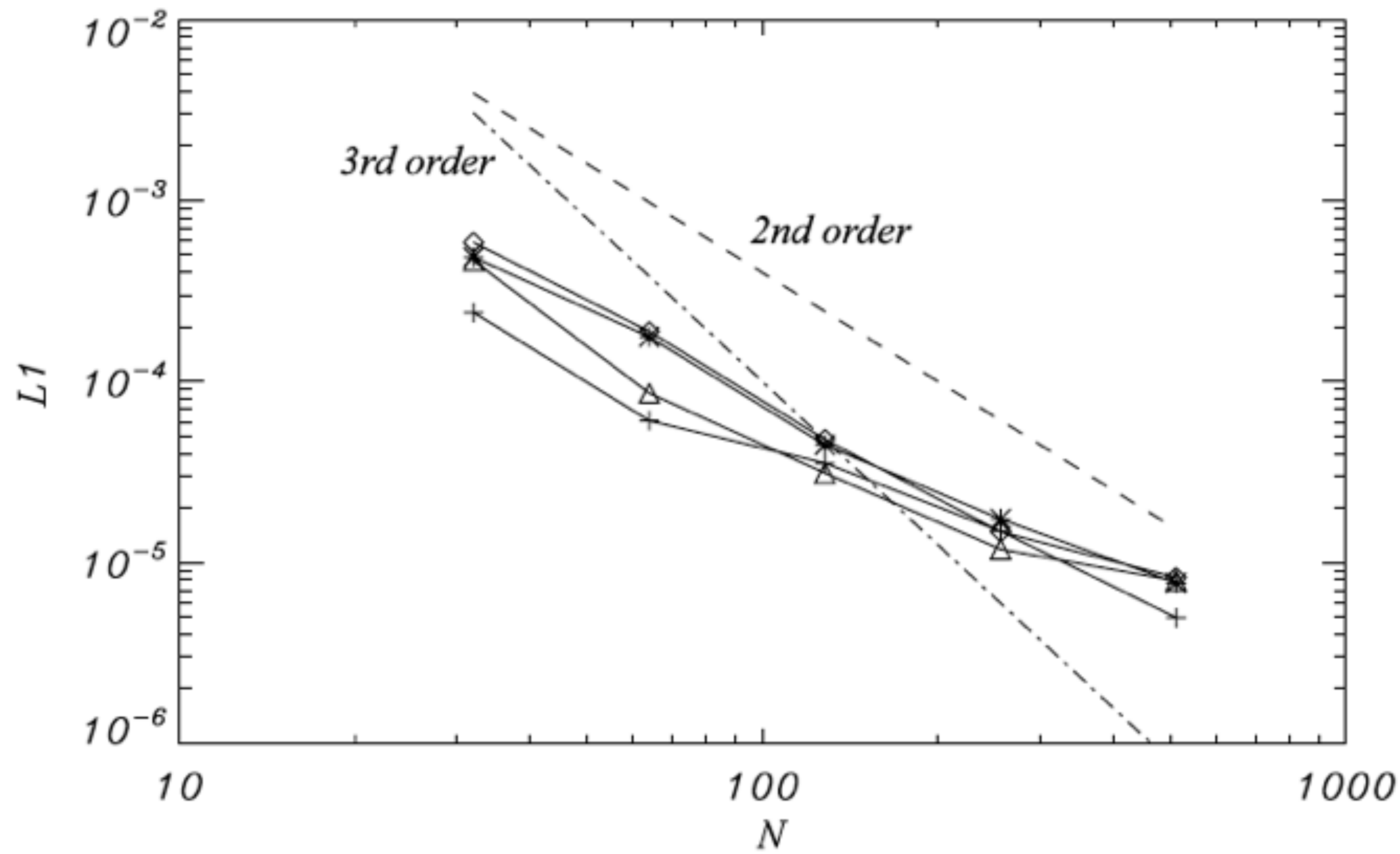
c/sqrt(3)

Numerical Tests

- Various set of numerical tests for validate the code accuracy and performance:
  - Wave propagation (comp. exact solution, check convergence)
  - Shock-Tube (comp. exact solution, check convergence)
  - Magnetic loop advection (check div. B problem)
  - blast wave propagation w./w.o. Magnetic field
  - Shock-shock interaction
  - Kelvin-Helmholtz instability (checking growth rate)
  - Jet propagation
  - Magnetic reconnection (checking resistivity)
  - etc.

Code Accuracy (L1 norm)

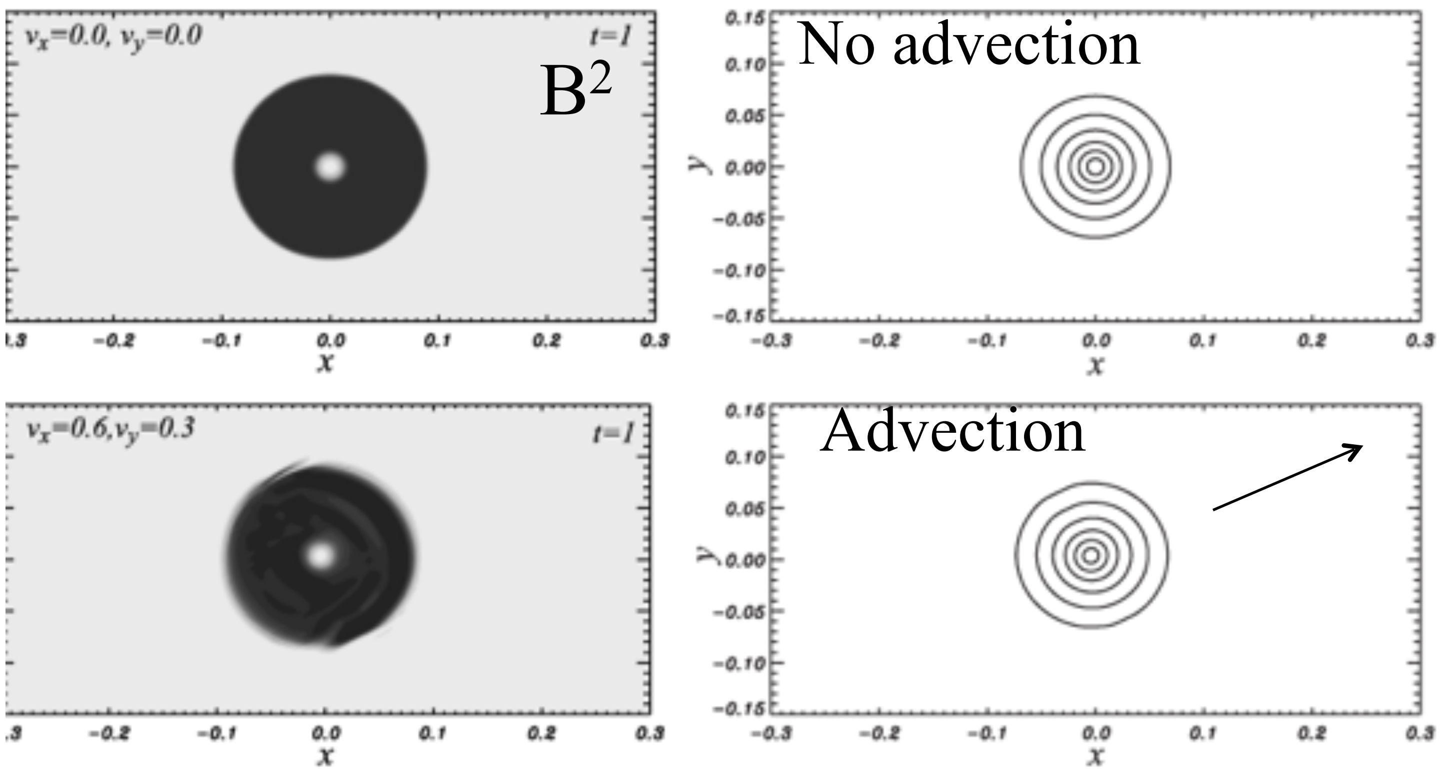
1D CP Alfven wave propagation test



L_1 norm errors of magnetic field v_y shows almost 2^{nd} order accuracy (RAISHIN code)

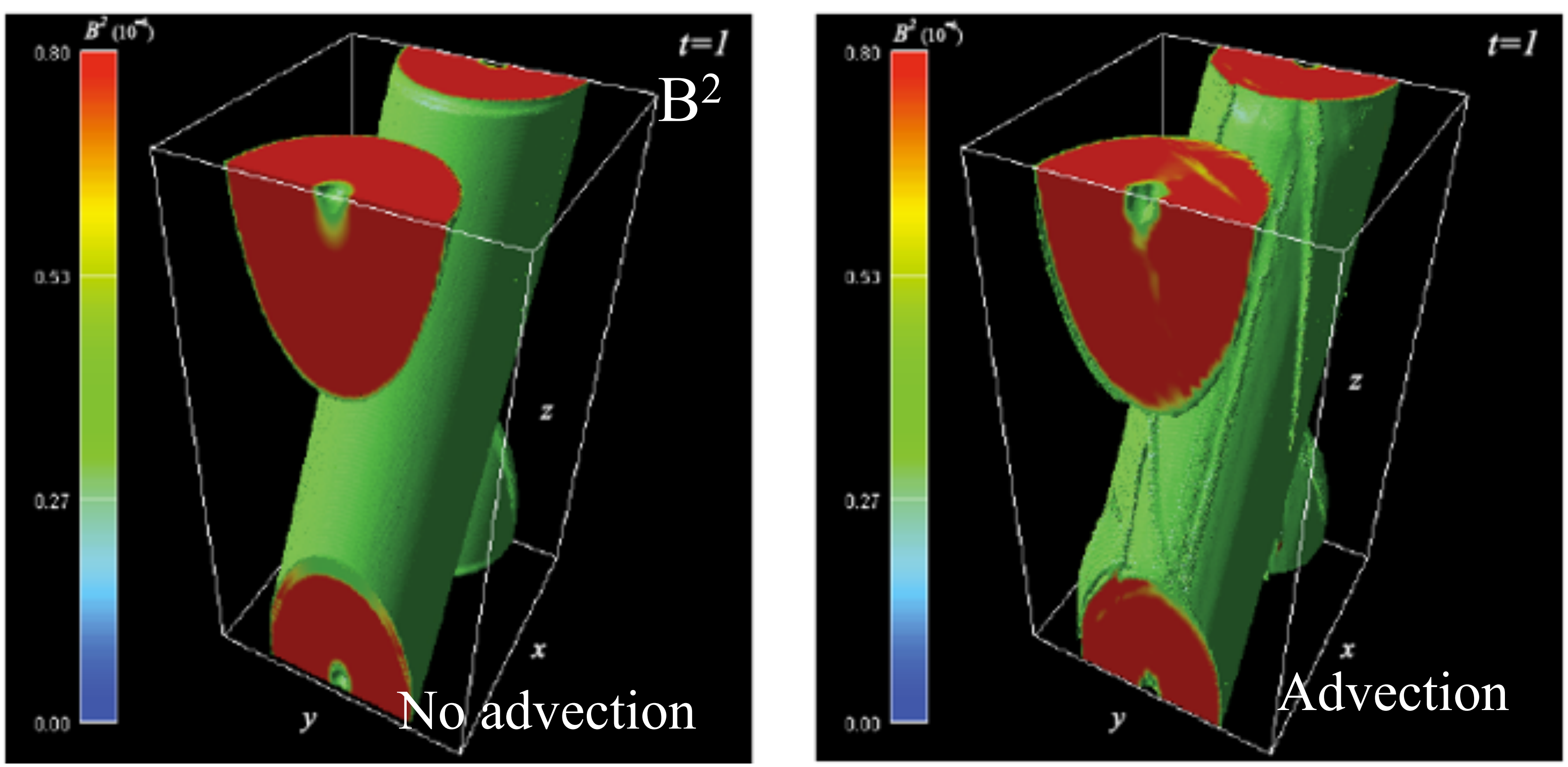
Advection of Magnetic Field Loop

2D

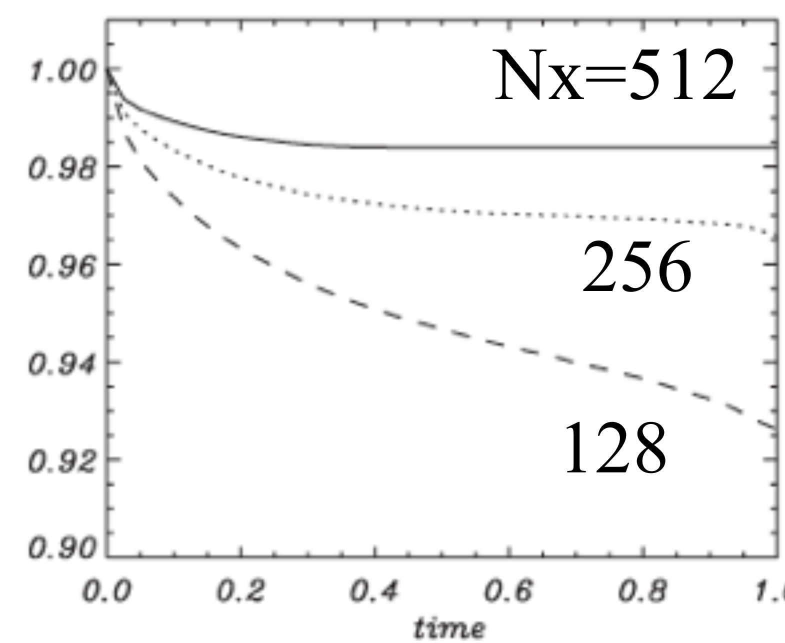


- Advection of a weak magnetic field loop in an uniform velocity field
- 2D: (v_x, v_y)=(0.6,0.3)
- 3D: (v_x,v_y,v_z)=(0.3,0.3,0.6)
- Periodic boundary in all direction
- Run until return to initial position in advection case

3D



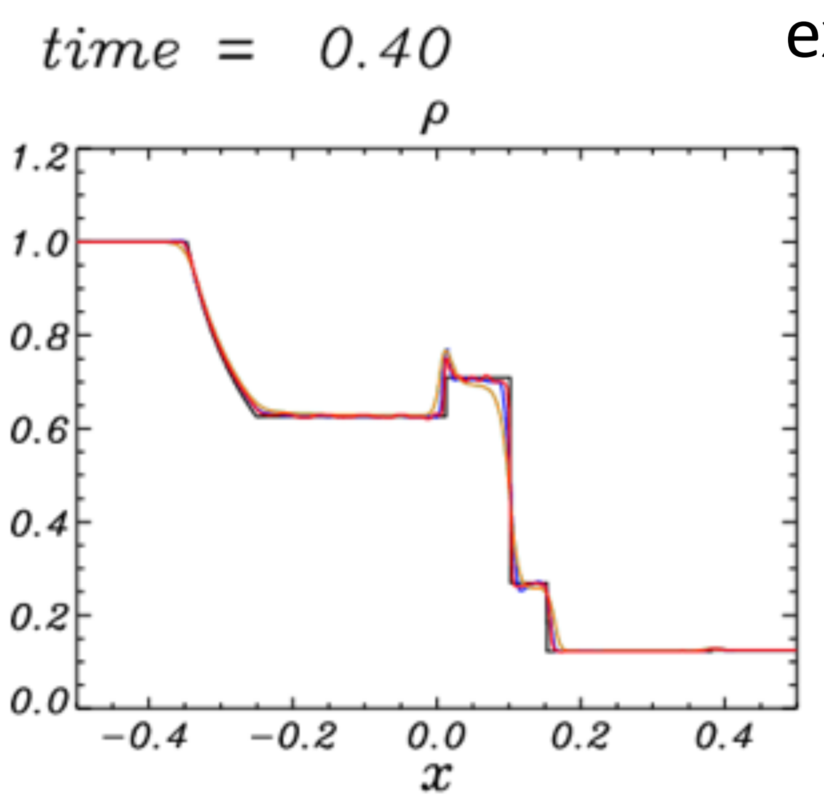
Volume-averaged magnetic energy density (2D)



Shock Propagation in High-Resolution Shock Capturing Scheme

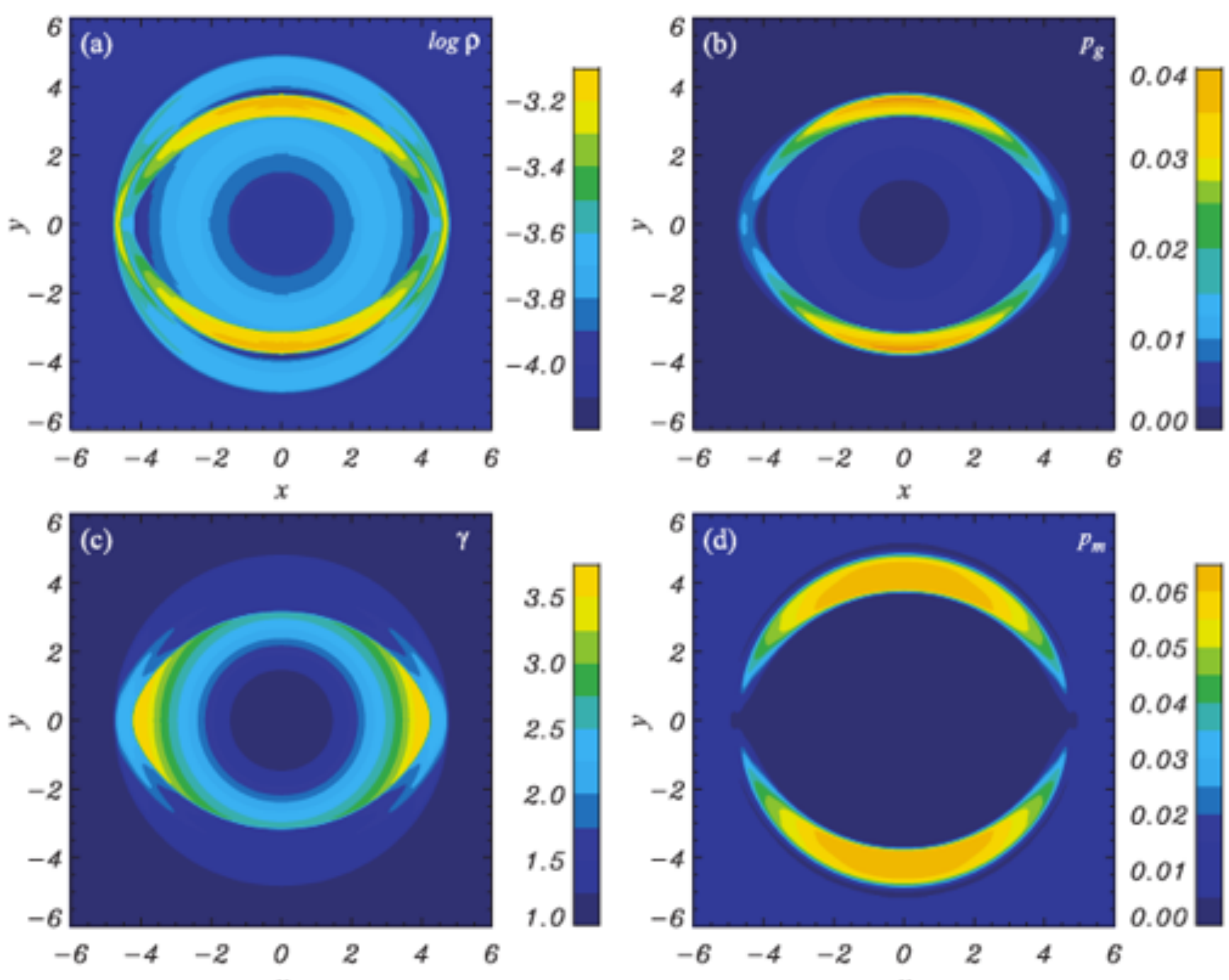
- Stable and accurate shock profiles
- Accurate propagation speed of discontinuities
- Accurate numerical resolution of nonlinear features: discontinuites, rarefaction waves, vortices, turbulence, etc

Shock-tube test



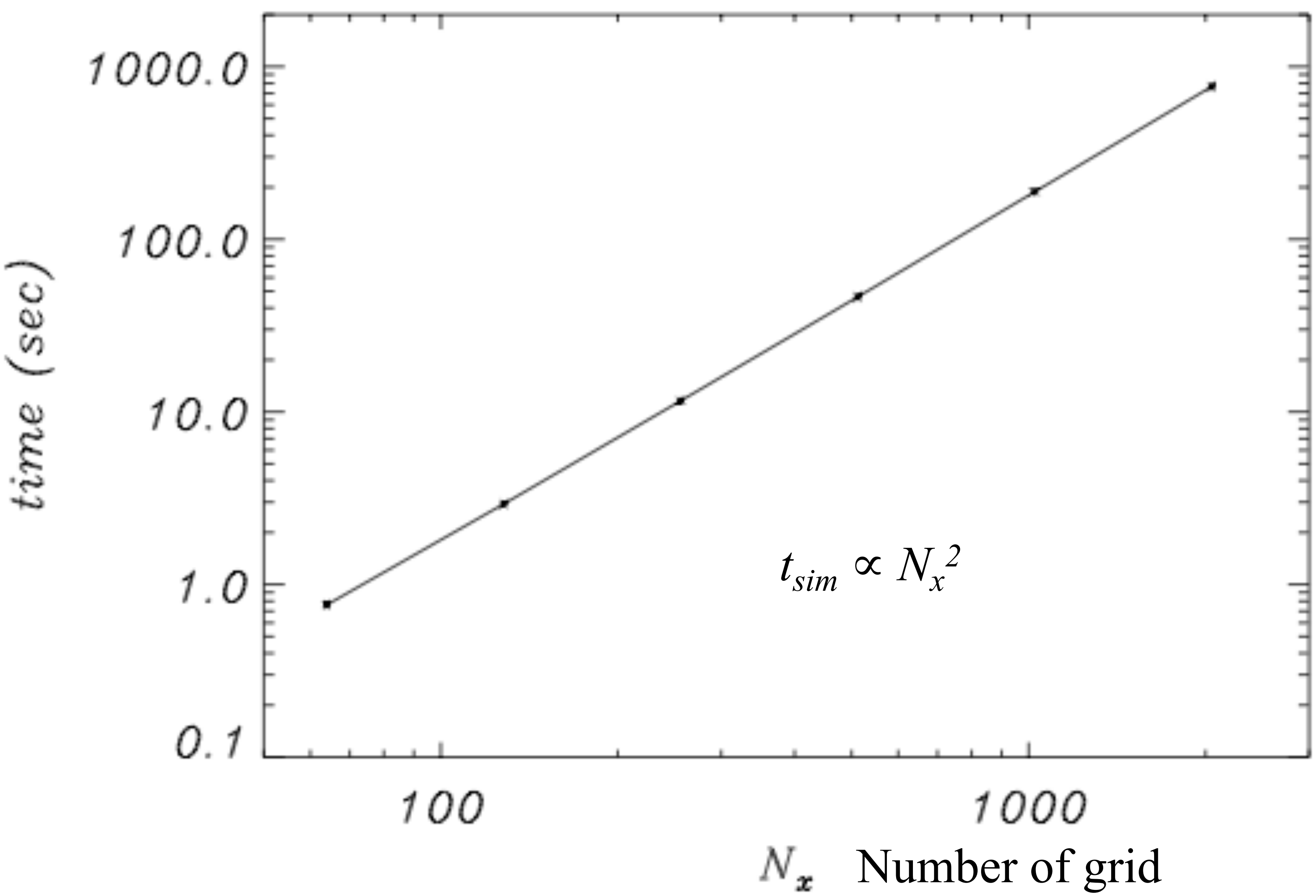
Black: exact solution, Blue: MC-limiter, Light blue: minmod-limiter, Orange: CENO, red: PPM

Cylindrical explosion



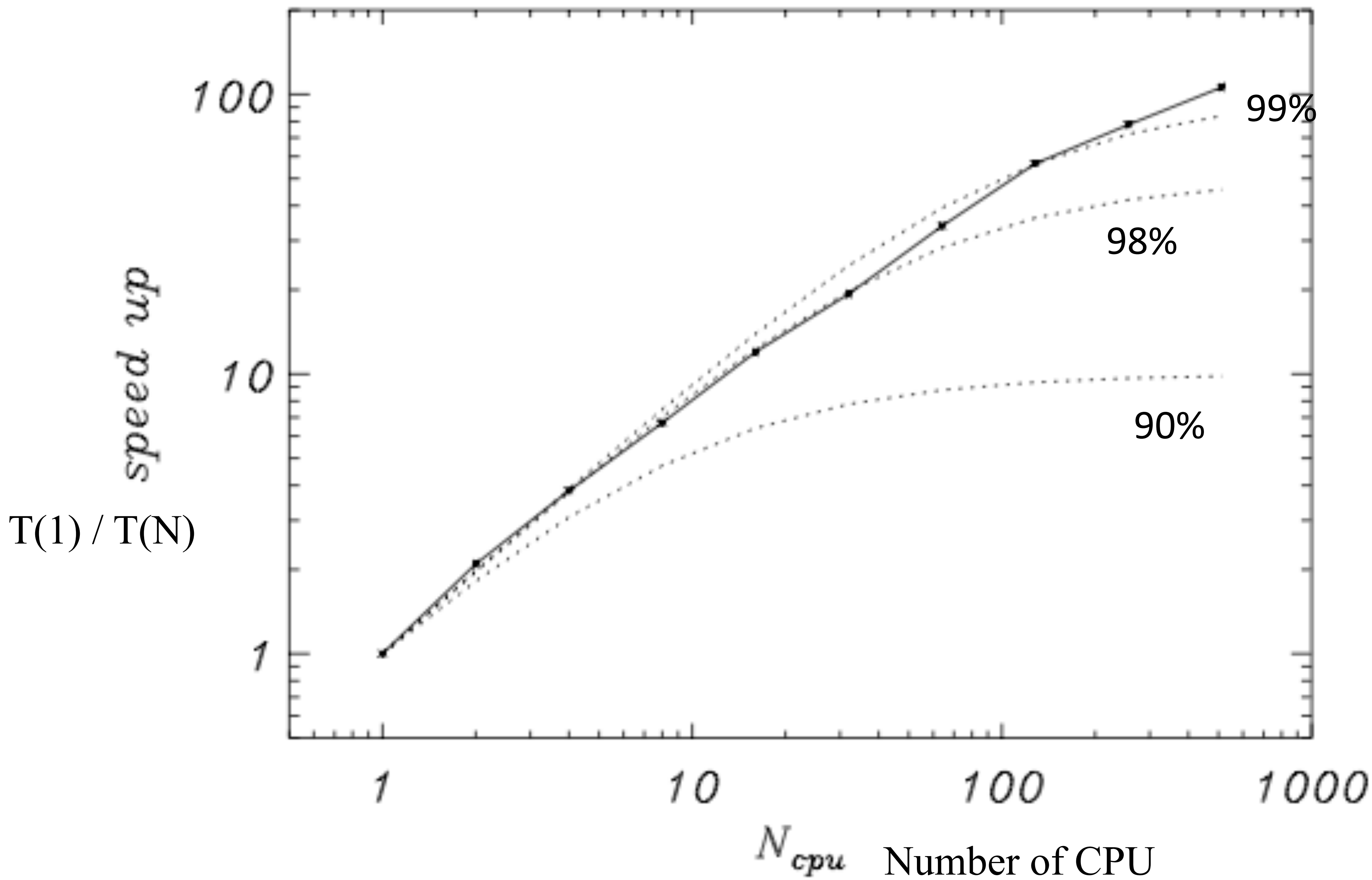
Code Accuracy
(grid number vs computer time)

1D shock-tube (Balsara Test 1) with 1 CPU calculated to t=0.4



Parallelization Accuracy

1D shock-tube (Balsara Test 1) in 3D Cartesian box (fixed grid number and size), calculated to t=0.4



Summary

- Finite-difference schemes are one of the commonly used numerical method for solving partial differential equations (PDEs) which are based on a discretization of the x-t plane with a mesh of discrete points (x_j, t^n)
- The simplest linear hyperbolic equation is the advection equation. And the simplest nonlinear hyperbolic equation is Burgers equation.
- Burgers equations show shock steepening which is a consequence of the propagation speeds not being constant.
- A Riemann problem is an initial value problem with discontinuous initial data. It consists of rarefaction waves, shock waves, and contact discontinuities.
- High resolution shock capturing scheme is high-order finite-difference (volume) methods solving conserved form of PDEs with appropriate amount of numerical dissipation in the vicinity of a discontinuity.