Lecture 3:

Essence of Special and General Relativity

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A Brief Review of Special Relativity

- Special relativity (SR) is the physical theory of measurement in inertial frames of reference proposed in 1905 by Albert Einstein

- It generalizes Galileo's principle of relativity – that all uniform motion is relative, and that there is no absolute and well-defined state of rest (no privileged reference frames) – from mechanics to all the laws of physics.

- In addition, special relativity incorporates the principle that the speed of light is the same for all inertial observers regardless of the state of motion of the source.

Simple Postulates

- The laws of physics are the same in every inertial frame of reference
- *The Principle of Relativity*

- The speed of light in vacuum is the same in all inertial frames of reference, and is independent of the motion of the source
- *Invariance of the speed of light*

Consequence of Special Relativity

- Time dilation: the time lapse between two events is not invariant from one observer to another, but is dependent on the relative speeds of the observers' reference frames.

- Relativity of simultaneity: two events happening in two different locations that occur simultaneously to one observer, may occur at different times to another observer (lack of absolute simultaneity).

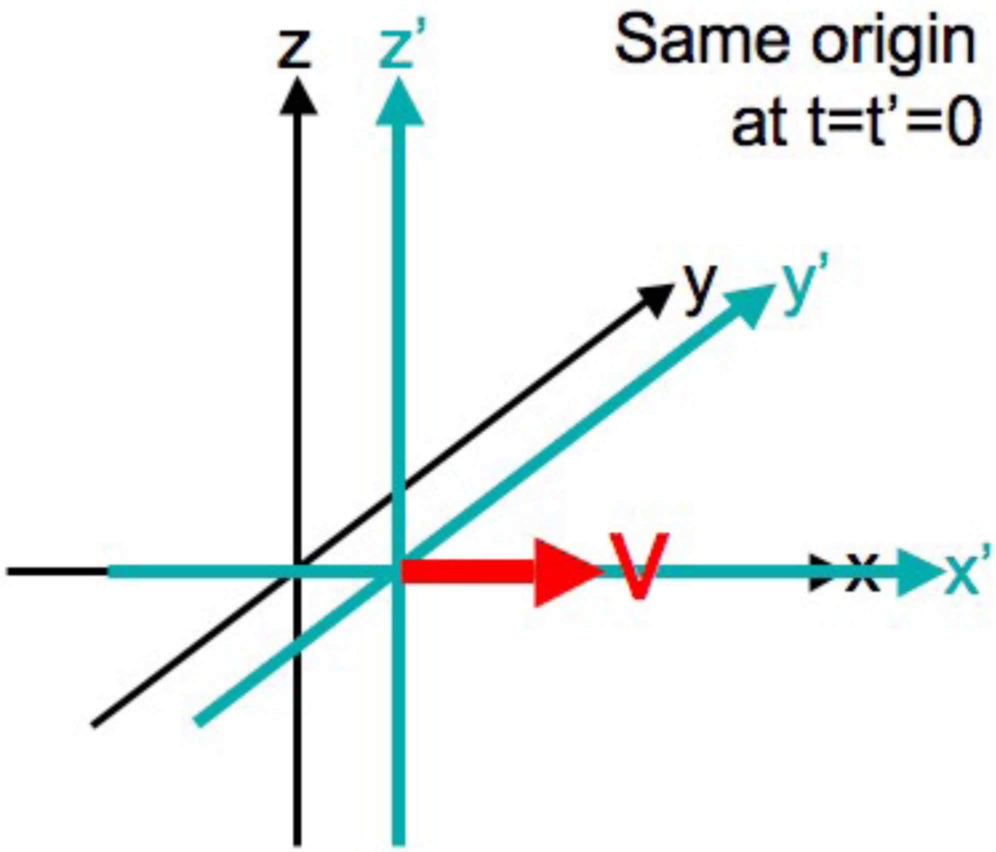
- Lorentz contraction: the dimensions (e.g., length) of an object as measured by one observer may be smaller than the results of measurements of the same object made by another observer.

- Composition of velocities: velocities (and speeds) do not simply 'add', for example if a rocket is moving at 2/3 the speed of light relative to an observer, and the rocket fires a missile at 2/3 of the speed of light relative to the rocket, the missile does not exceed the speed of light relative to the observer.

- Inertia and momentum: as an object's speed approaches the speed of light from an observer's point of view, its mass appears to increase thereby making it more and more difficult to accelerate it from within the observer's frame of reference.

- Equivalence of mass and energy, $E = mc^2$: Conservation of energy implies that in any reaction a decrease of the sum of the masses of particles must be accompanied by an increase in kinetic energies of the particles after the reaction.

Lorentz Transformations



$$\begin{cases} t' = \gamma(t - vx) \\ x' = \gamma(x - vt) \\ y' = y \\ z' = z \end{cases}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

vector notation for events

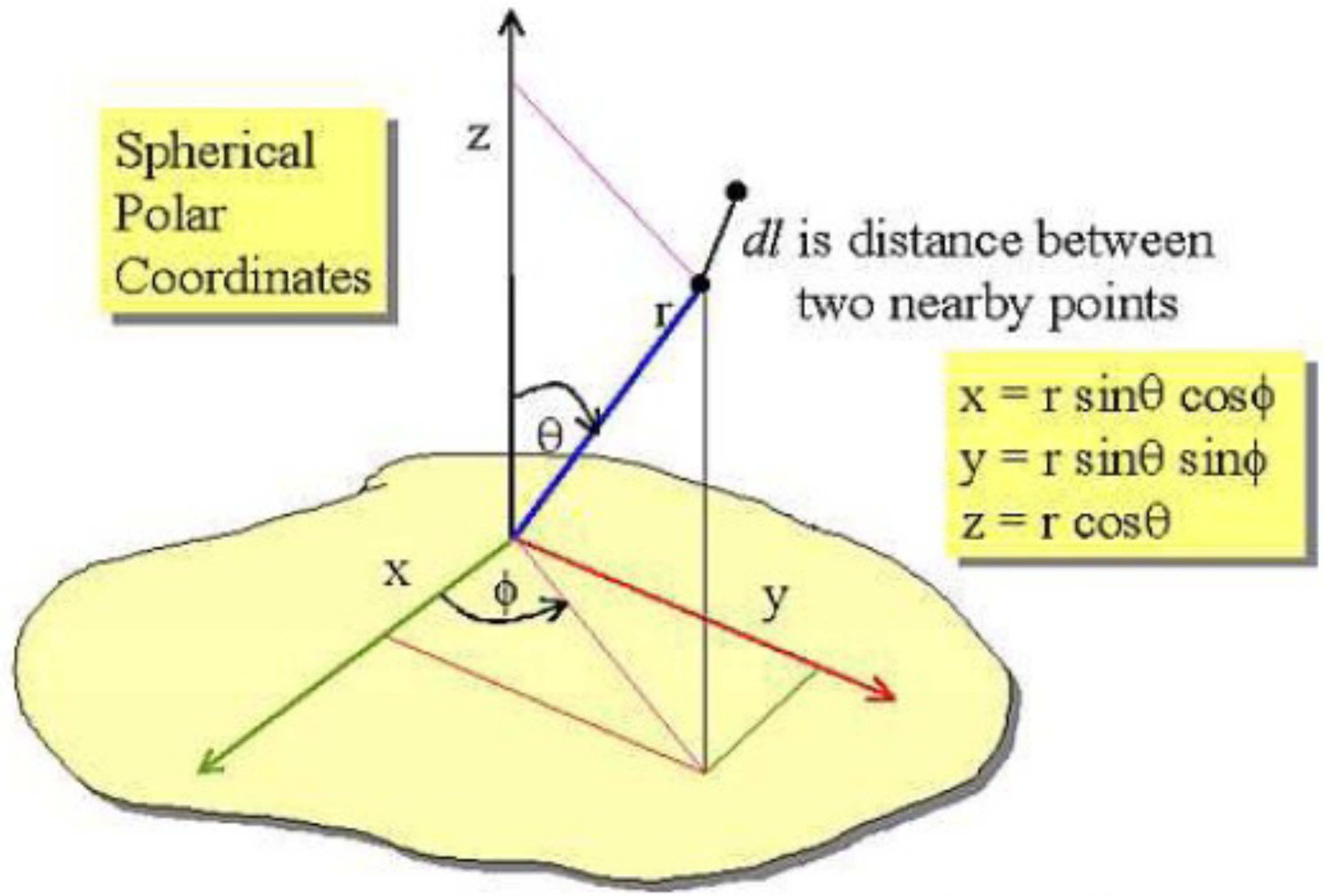
$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$x'^\mu = \Lambda^\mu_\nu x^\nu$$

Minkowski line element

flat spacetime
= Minkowski spacetime

$$ds^2 = -c^2dt^2 + dl^2$$



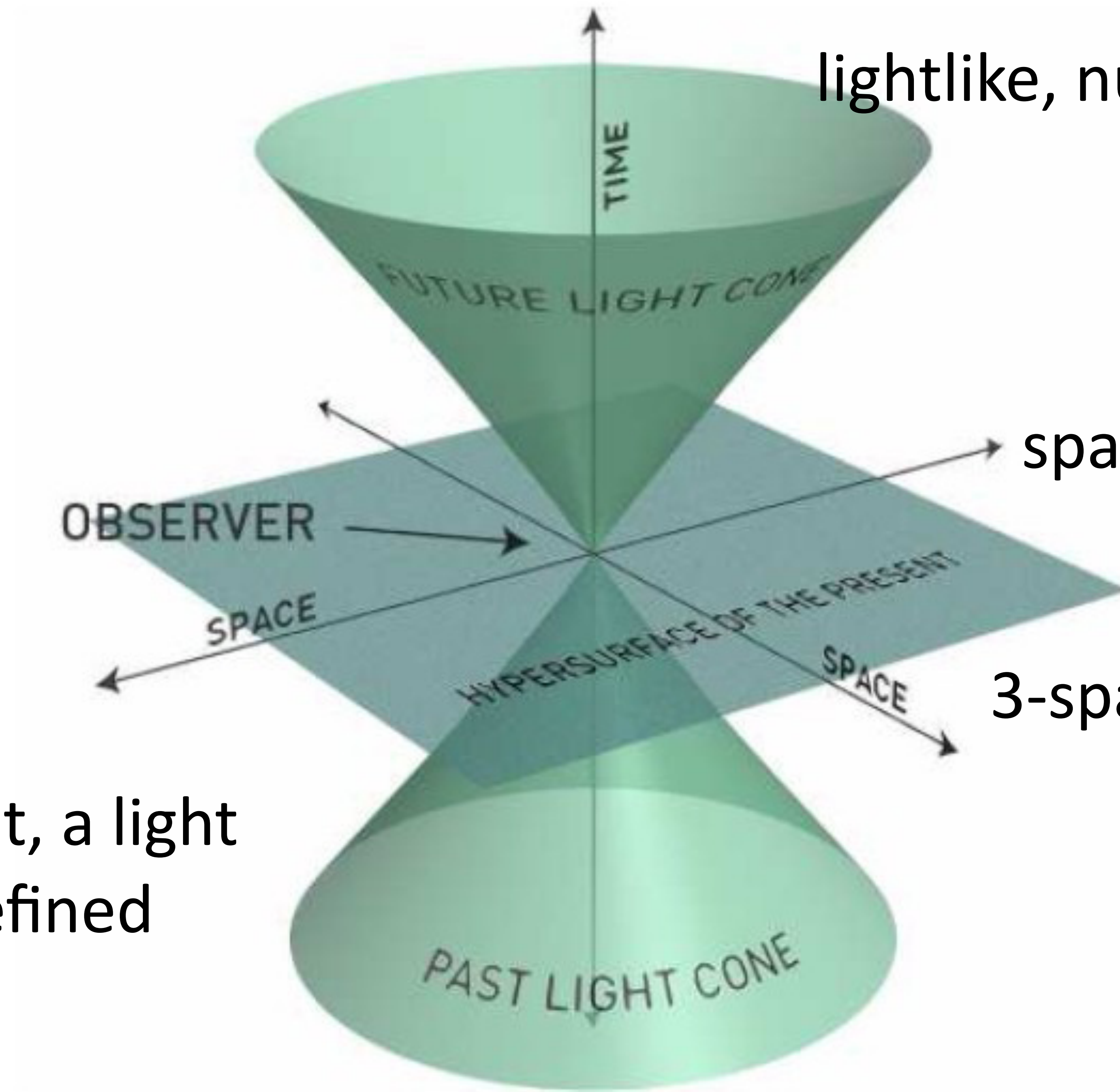
$$dl^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

event: (ct, x^i) => spacetime (coupling with time & space)
= set of all events

Causal Structure of Spacetime

timelike

lightlike, null



spacelike

3-space

At each event, a light cone is defined

Experimental Tests of SR

- SR makes many predictions, which are well tested:
  - Isotropy of the speed of light
  - Isotropy of space
  - Constancy of the speed of light
  - Time dilation and Doppler
  - Length contraction
  - Twin paradox
  - Relativistic kinematics
  - Relativistic velocity addition
  - Variation of c with frequency
  - g-2 as test of SR
  - Other – 14 experiments!

- The isotropy of c is particularly well tested:
  - Michelson-Morley (and variations), Laser/Maser tests, Atomic beams, Frequency-doubling interferometer, Cryogenic optical resonators etc.

The Laws of Physics in SR

- Write them as tensor equations (tensors are Lorentz covariant entities).
- E and B fields in Maxwell's theory e.g. are not covariant => use Faraday tensor.
- Use conservation of energy and momentum.
- Derive field equations, if possible, from Lagrangians (for microscopic theories).

Summary of Special Relativity

- Special Relativity (1905) is well established.
- Invariance of the speed of light is well tested, no preferred frame of reference in Minkowski (flat spacetime)
- Laws of physics are to be written in covariant way:
  - Maxwell's theory with Faraday tensor etc.
- Question > 1905: How to include gravity?

From SR to GR

- Essential elements of SR are only local concepts:
  (i) Concept of Minkowski spacetime
  (ii) Concept of a metric $\boldsymbol{g}$ for distance measurements => notion of geodesics
  (iii) Causal structure of SR induced by metric $\boldsymbol{g}$ => locally Minkowskian
- How to incorporate gravity?
  => via Equivalence Principles => Geodesic => Spacetime as manifold of events
- GR: Ricci tensor couples to all types of matter, but many other metric theories.

The equivalence principle

- The equivalence principle in few common words: "All things fall in the same way".
- Or slightly more wordy: "all objects have the same acceleration in a gravitational field" (e.g. a feather and bowling ball fall with the same acceleration in the absence of air friction).
- The fact that "All things fall in the same way" is true because the "inertial" mass that enters Newton's law of motion, $F=ma$, is the same as the "gravitational" mass that enters the gravitational-force law.
- The principle of equivalence is really a statement that inertial and gravitational masses are equal to each other for any object.

The equivalence principle

- Einstein's equivalence principle:
  - Uniqueness of free fall
  - Local Lorentz invariance
  - Local Position invariance

- Metric theory: definition in GR
  - Curved spacetime (Riemann manifold) is endowed with a symmetric four-dimensional metric
  - Trajectory of free falling bodies are geodesics of that metric
- Einstein equivalence principle => Only metric theory viable.

General Relativity and The Equivalence Principle

- Einstein started to think of the path of an object as a property of spacetime itself, rather than being related with the specific properties of the object.

- The idea is that gravity is a manifestation of the fact that objects in free fall follow geodesics in curved spacetimes.

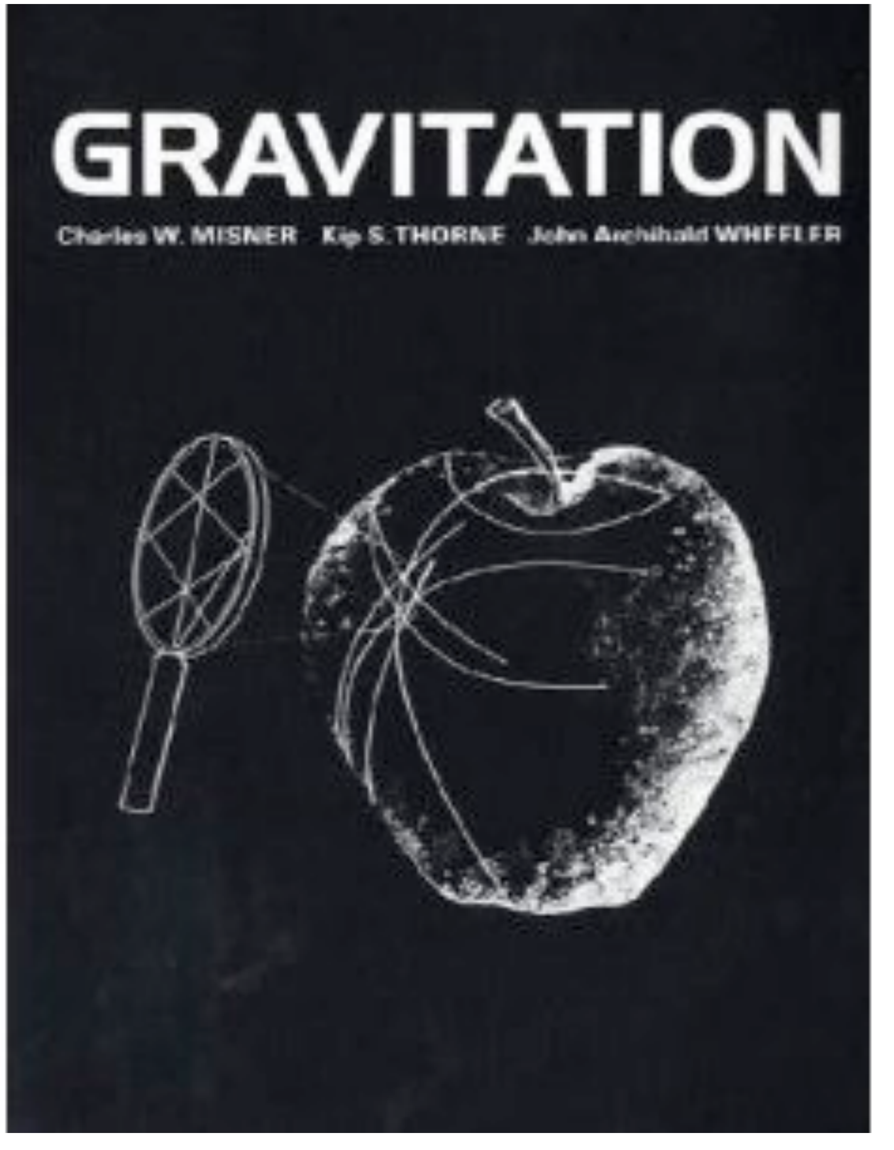
- What are geodesics?

- We know in our ordinary experience (flat Minkowski spacetime) that in the absence of any forces, objects follow straight lines, and we also know that straight lines are the shortest possible paths that connect two points in such conditions.

- The generalization of the notion of a "straight line" valid also in curved spacetimes (Riemann manifold) is called geodesic.

Ants on the Apple

- A famous story to simply illustrate the idea of general relativity is the "Parable of the Apple" by Misner, Thorne, and Wheeler [Gravitation (1973)].
- The parable tries to explain the nature of gravitation in terms of the curvature of spacetime.
- The spacetime of the parable is the two-dimensional curved surface of an apple.



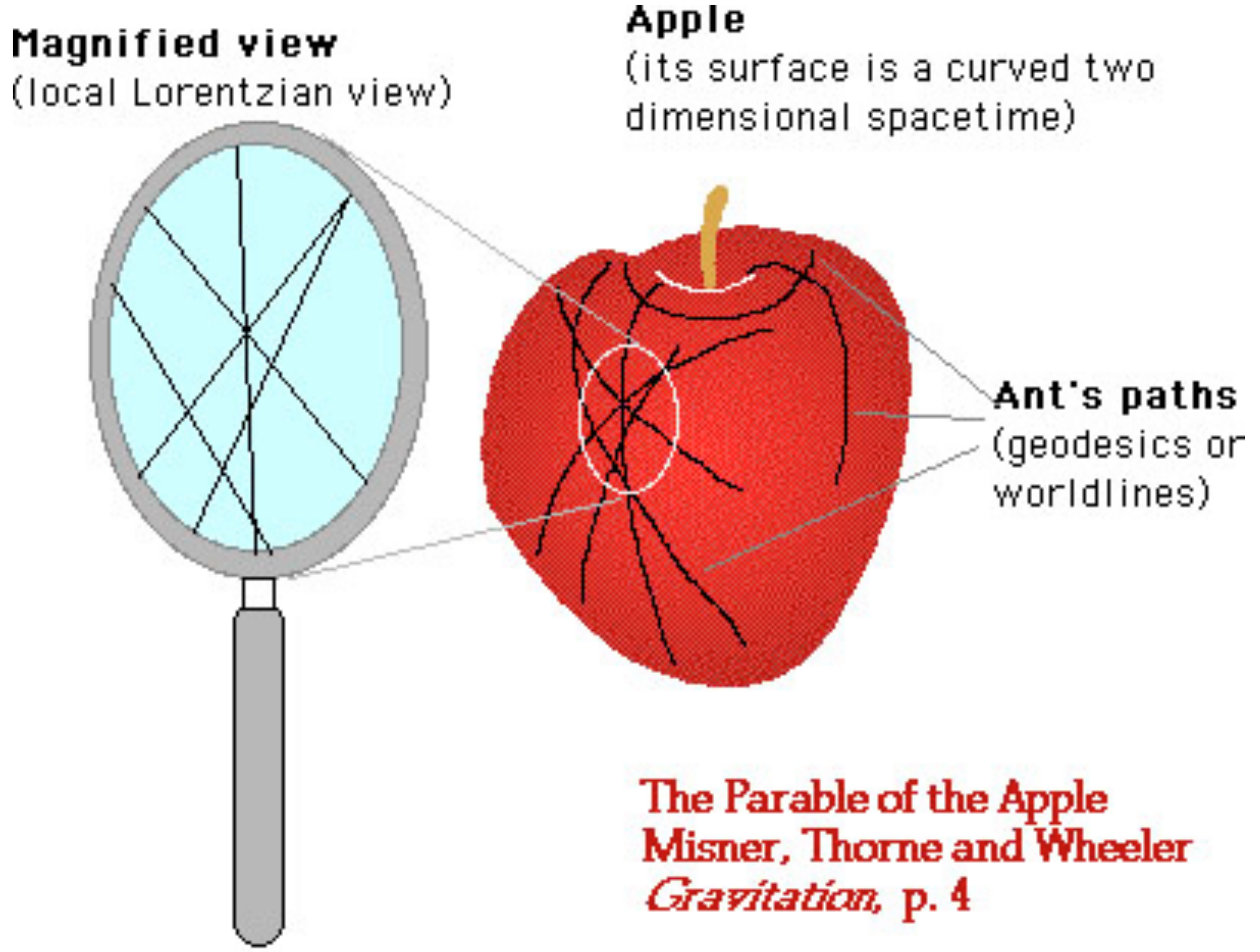
Ants on the Apple

- The tale goes like this. One day a student, reflecting on the difference between Einstein's and Newton's views about gravity, noticed ants running on the surface of an apple.

- By advancing alternately and of the same amount the left and right legs, the ants seemed to take the most economical path; "wow, they are going along geodesics on this surface!"

- The student followed the path of an ant tracing it and then cutting with a knife a small stripe around the trace: "Indeed when put on a plane the path is a straight line!"

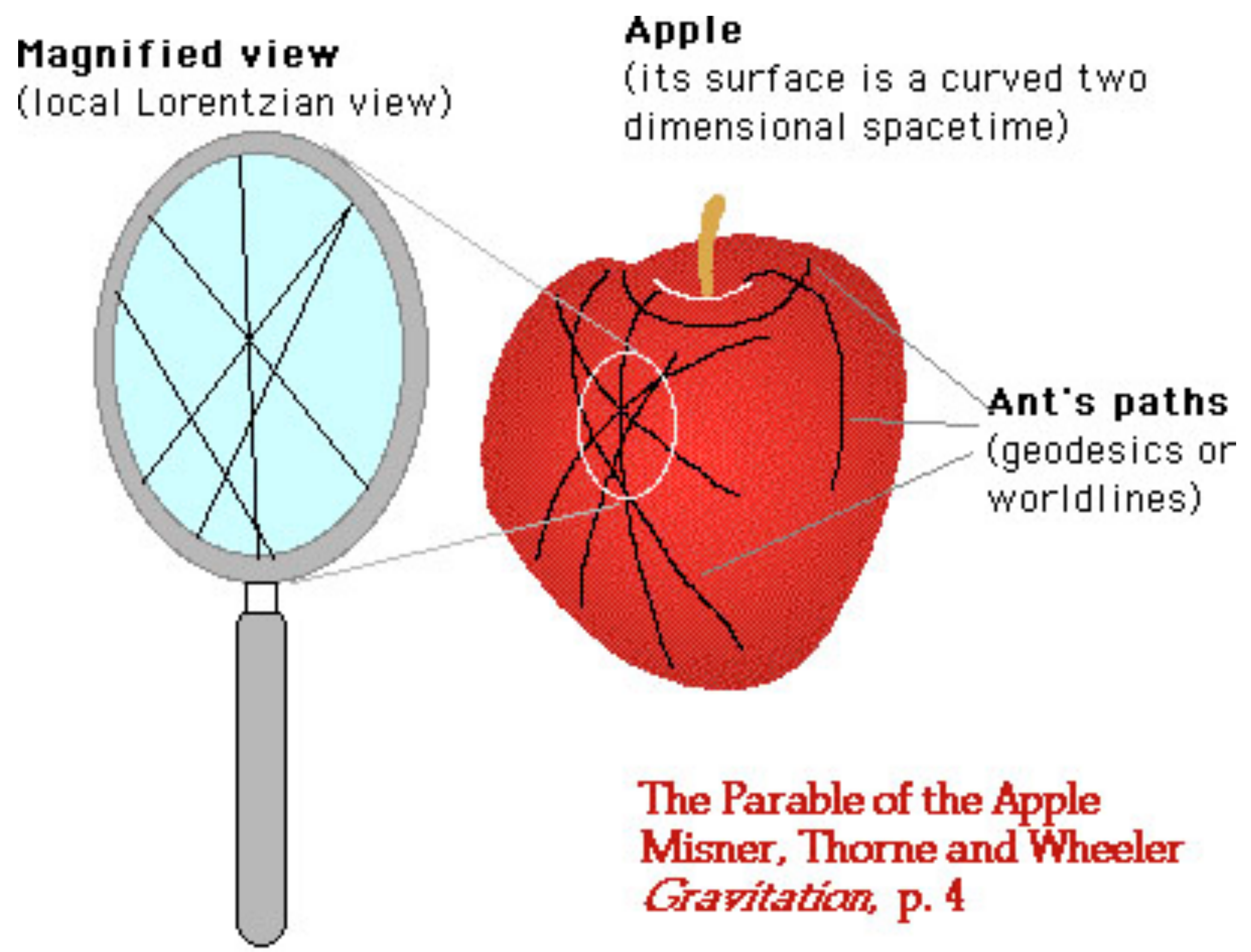
- Each geodesic may be regarded as a path (world line) of a free particle on this surface (taken as a two-dimensional spacetime).



The Parable of the Apple
Misner, Thorne and Wheeler
Gravitation, p. 4

Ants on the Apple

- Then the student looked at two ants going from the same spot onto initially divergent paths, but then, when approaching the top (near the dimple) of the apple, the paths crossed and continued into different directions!
- The reason of the curved trajectories is:
- According to Newton, gravitation is acting at a distance from a center of attraction (the dimple).
- According to Einstein, the local geometry of the surface around the dimple is curved.



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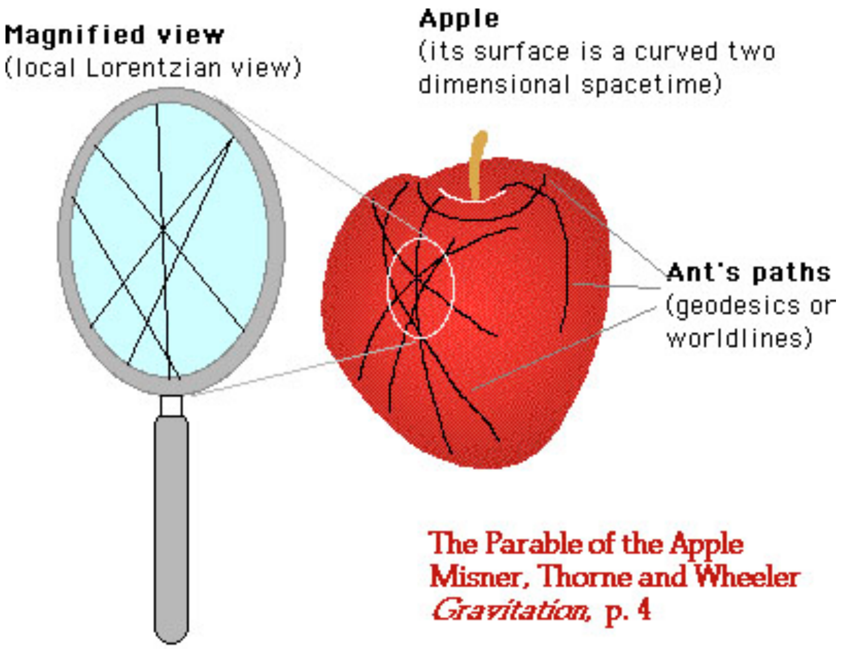
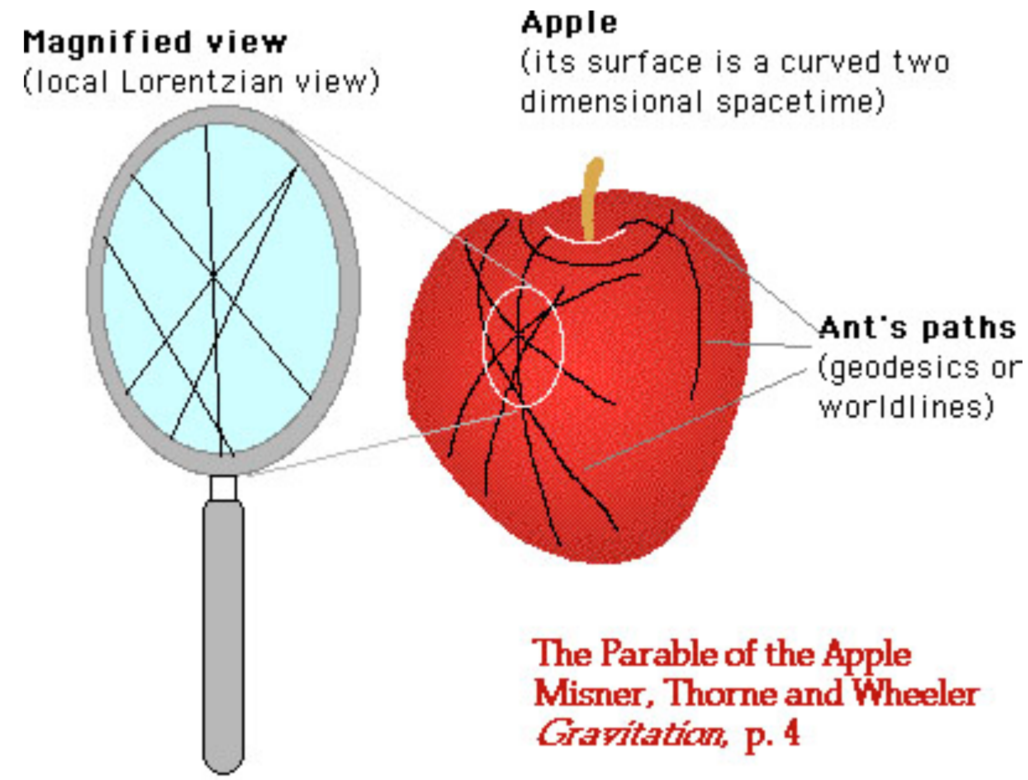
Ants on the Apple



- Comments:
  - Einstein interpretation dispenses with any action-at-a-distance.
  - Although the surface of the apple is curved, if you look at any local spot closely (with a magnifying glass), its geometry looks like that of a flat surface (the Minkowski spacetime of SR).
  - The interaction of spacetime and matter is summarized in Wheeler's favorite words, “*spacetime tells matter how to move, and matter tells spacetime how to curve*”.
  - This reciprocal influence (matter <=> spacetime) makes Einstein’s field equation non-linear and so very hard to solve.

Ants on the Apple



Summary of the parable:

1) objects follow geodesics and locally geodesics appear straight

2) over more extended regions of space and time, geodesics originally receding from each other begin to approach at a rate governed by the curvature of spacetime, and this effect of geometry on matter is what was called "gravitation"

3) matter in turn warps geometry.

Einstein Equations and General Relativity

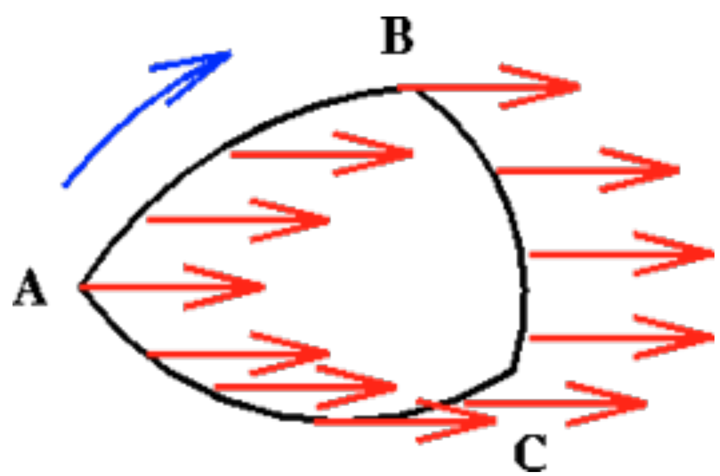
Important quantities in general relativity:

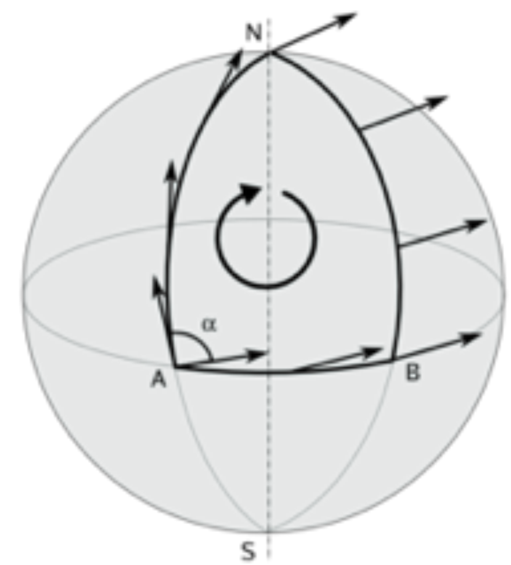
- the metric (the metric tensor g), which may be regarded as a machinery for measuring distances: $$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

- curvature, expressed by the Riemann curvature tensor

$$R^\alpha_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu}$$

(where $\Gamma^\alpha_{\beta\mu} = \frac{1}{2}g^{\alpha\sigma}\left(g_{\beta\sigma,\mu} + g_{\sigma\mu,\beta} - g_{\beta\mu,\sigma}\right)$ are Christoffel symbols)





- the Ricci tensor $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$ and the curvature (Ricci) scalar $R = g^{\mu\nu}R_{\mu\nu}$
- covariant derivative: a derivative that takes into account the curvature of the spacetime

Einstein Equations and General Relativity

- From these quantities the path of any particle can be calculated. This is how "*geometry tells matter how to move*".

- The other direction ("*matter tells spacetime how to curve*") requires to know the distribution of matter (mass/energy/momentum), described through the stress-energy tensor $\boldsymbol{T}$.

- After many years of thinking, Einstein reached a satisfactory form for the equations relating geometry and matter:

$$\text{Einstein\_tensor} = \text{const. x } \boldsymbol{T}$$

- The Einstein tensor (usually called $\boldsymbol{G}$) is a tensor in 4D spacetime that has the wanted properties of:
  - being a symmetric tensor (it must because the stress-energy tensor is symmetric)
  - having vanishing (covariant) divergence (it must because the stress-energy tensor has vanishing divergence)
  - the weak-field limit of the Einstein equations gives the Newtonian Poisson equation (from the comparison to which the value of the above constant is found)

Einstein's Field Equations

Einstein tensor (spacetime) | Ricci tensor | curvature scalar | matter and other fields

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

metric (measure of spacetime distance)

simplified form, no cosmological constant

Energy-momentum tensor | 4-velocity

$$T^{\mu\nu} = \rho(1 + \epsilon + p/\rho) u^\mu u^\nu + p g^{\mu\nu}$$

rest-mass density | internal energy density | pressure

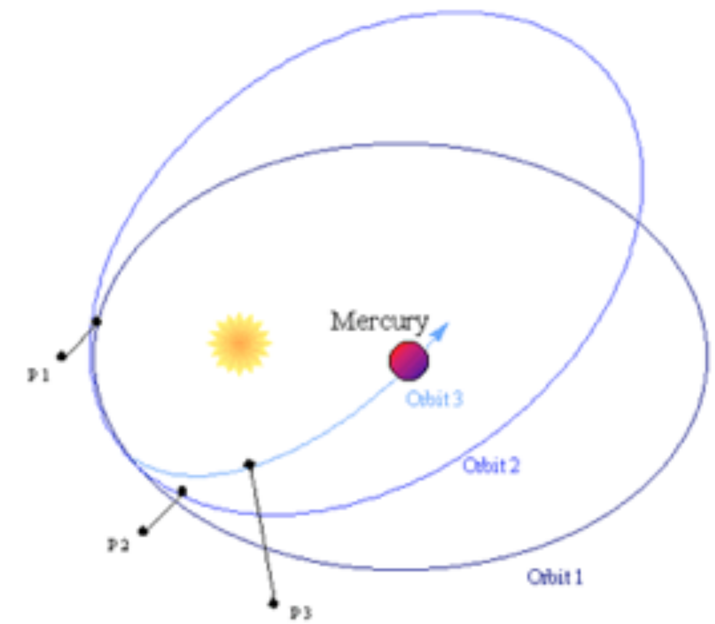
Trajectories of freely falling bodies are geodesics

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0$$

Geodesic equation

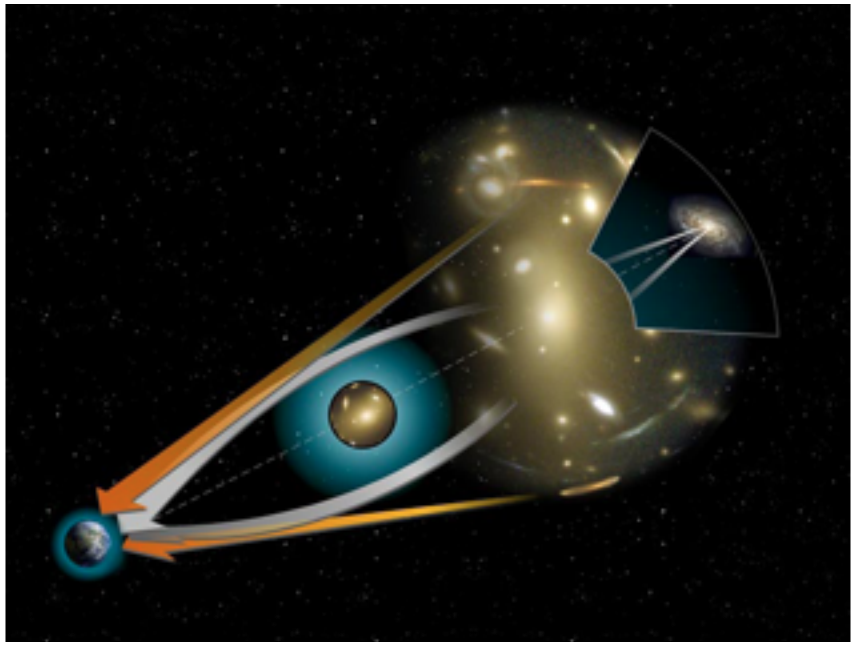
Some verified prediction of GR

- Gravitational redshift
  - change in the frequency of light moving in regions of different curvature
  - measured in experiments and taken into account by the GPS



- Periastron shift
  - first measured in the perihelion advance of Mercury (the GR prediction coincides with the "anomalous" advance, if computed in Newtonian theory)
  - binary pulsar (strong fields, so larger shifts)

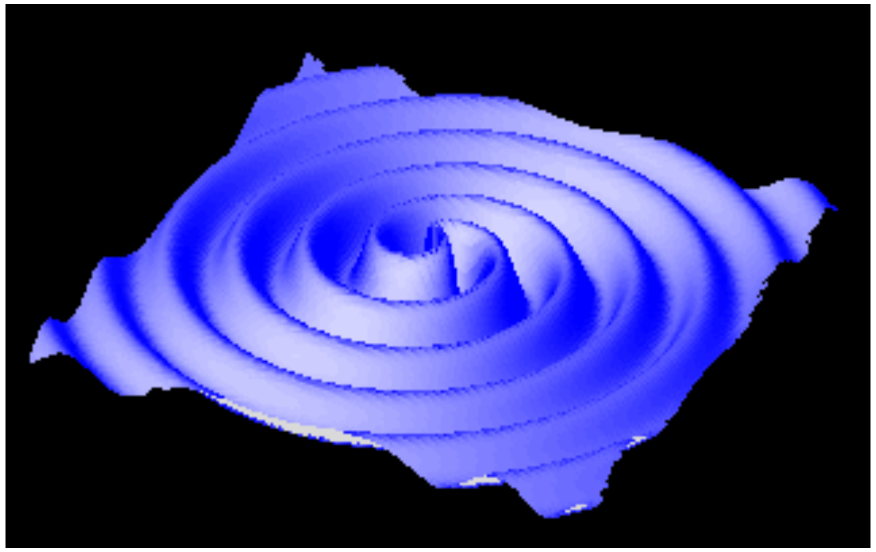
- Bending of light
  - first measured in the bending of photons traveling near the Sun
  - gravitational lensing



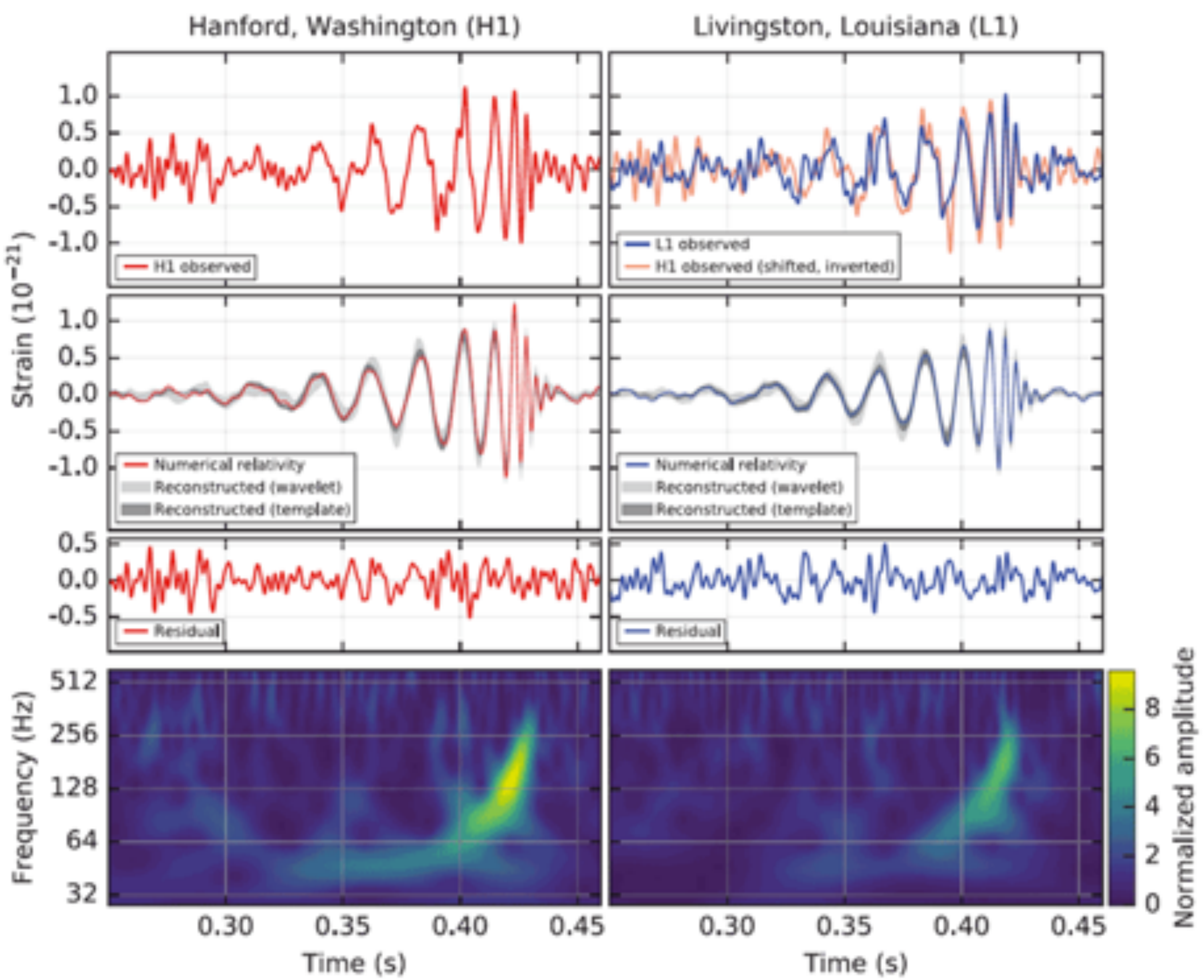
Prediction of General Relativity: Black Holes

- A black hole is literally a region from which light cannot escape.
- Black Hole is a stationary solution of Einstein's vacuum equations $R_{\alpha\beta} = 0$, in which the gravitational field is so powerful that nothing, not even electromagnetic radiation (e.g. visible light), can escape its pull after having fallen past its event horizon.
- The term derives from the fact that the absorption of visible light renders the hole's interior invisible, and indistinguishable from the black space around it.
- Despite its interior being invisible, a black hole may reveal its presence through an interaction with matter that lies in orbit outside its event horizon => Black Hole shadow
- There are several candidate objects that are thought to be black holes (because they are very compact), but there has been no direct observation of black holes in electromagnetic waves up to now (recently detected the gravitational waves by BH-BH merger).

Prediction of General Relativity: Gravitational Wave



- In general relativity, disturbances in the spacetime curvature (the “old” “gravitational field”) propagate at the speed of light as gravitational radiation or gravitational waves (also known as gravity waves, but this term was already in use in fluid dynamics with a different meaning, so I recommend to avoid it) .
- Gravitational waves are a strong point of general relativity, which solves the action-at-distance problem of Newtonian gravity.
- Analogously to light, which is produced by the motion of electric charges, gravitational waves are produced by the motion of ... anything (mass-energy). When you wave your hand, you make gravitational waves.
- The point is that the amplitude of gravitational radiation is very small.



Birkhoff’s Theorem

- Birkhoff's theorem states that in GR, any spherically symmetric solution of the vacuum field equations must be stationary and asymptotically flat.
- This means that the exterior solution must be given by the Schwarzschild metric.
- This means that all spherical gravitational fields, whether from a star or from a black hole, are indistinguishable at large distances.
- A consequence of this is that purely radial changes in a spherical star do not affect its external gravitational field => no scalar modes of gravitational waves

Schwarzschild Black Hole

Schwarzschild BH is stationary symmetric solution of Einstein's vacuum field equations. This BH's physical properties are only mass.

Schwarzschild metric (in Boyer-Lindquist coordinates)

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

$r_s = \dfrac{2GM}{c^2}$: Schwarzschild radius (= horizon radius)

If $r \to \infty$ or $M = 0$

Schwarzschild metric goes to normal flat Minkowski spacetime (asymptotically flat)

Schwarzschild Black Hole

- In Schwarzschild metric (in Boyer-Lindquist coordinates) $\mathrm{r = 2GM/c^2}$ is a coordinate singularity, surface of infinite redshift => Event Horizon.
- Exterior Schwarzschild solution can be extended inside horizon with some special coordinates as true vacuum solution.
  - The metric is well-behaved at Schwarzschild radius
- r=0 is a true curvature singularity.
- This singularity is however hidden behind a horizon (Penrose suggested the "Cosmic Censorship" hypothesis).
- Horizon is no particular surface => you can move with your spaceship through the horizon and tell us what happens inside?
- Matter is not important for BH solution.

Orbits in Schwarzschild BH

- The equations governing the geodesics in a spacetime can be derived from the energy integral given by the Lagrangian

$$2\mathcal{L} = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$$

- where λ is some affine parameter along the geodesic. For time-like geodesics, λ may be identified with the proper time τ.

$$\mathcal{L} = \frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1-2M/r} - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2\right]$$

- Calculate the respective canonical momenta as $p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}$

$$p_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = \left(1-\frac{2M}{r}\right)\dot{t} \qquad p_r = -\frac{\partial \mathcal{L}}{\partial \dot{r}} = \left(1-\frac{2M}{r}\right)^{-1}\dot{r}$$

$$p_\theta = -\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = r^2\dot{\theta} \qquad p_\phi = -\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = (r^2\sin^2\theta)\dot{\phi}$$

Integrals of Motion

Further integrals of motion follow from the equations

$$\frac{dp_t}{d\tau} = \frac{\partial \mathcal{L}}{\partial t} = 0$$

$$\frac{dp_\phi}{d\tau} = \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

Thus we find

$$p_t = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = E = \text{ const.}$$ Energy Integral

and

$$p_\phi = r^2 \sin^2\theta \frac{d\phi}{d\tau} = L = \text{ const.}$$ Angular momentum

Effective Potential

$$P_\phi = r^2 \frac{d\phi}{d\tau} = L = \text{const.}$$ equatorial plane

rewrite Lagrangian

$$\frac{E^2}{1-2M/r} - \frac{\dot{r}^2}{1-2M/r} - \frac{L^2}{r^2} = 2\mathcal{L} = 1, 0$$

massive particle or
massless particle

find two integrals of motion

$$\left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right) = E^2$$

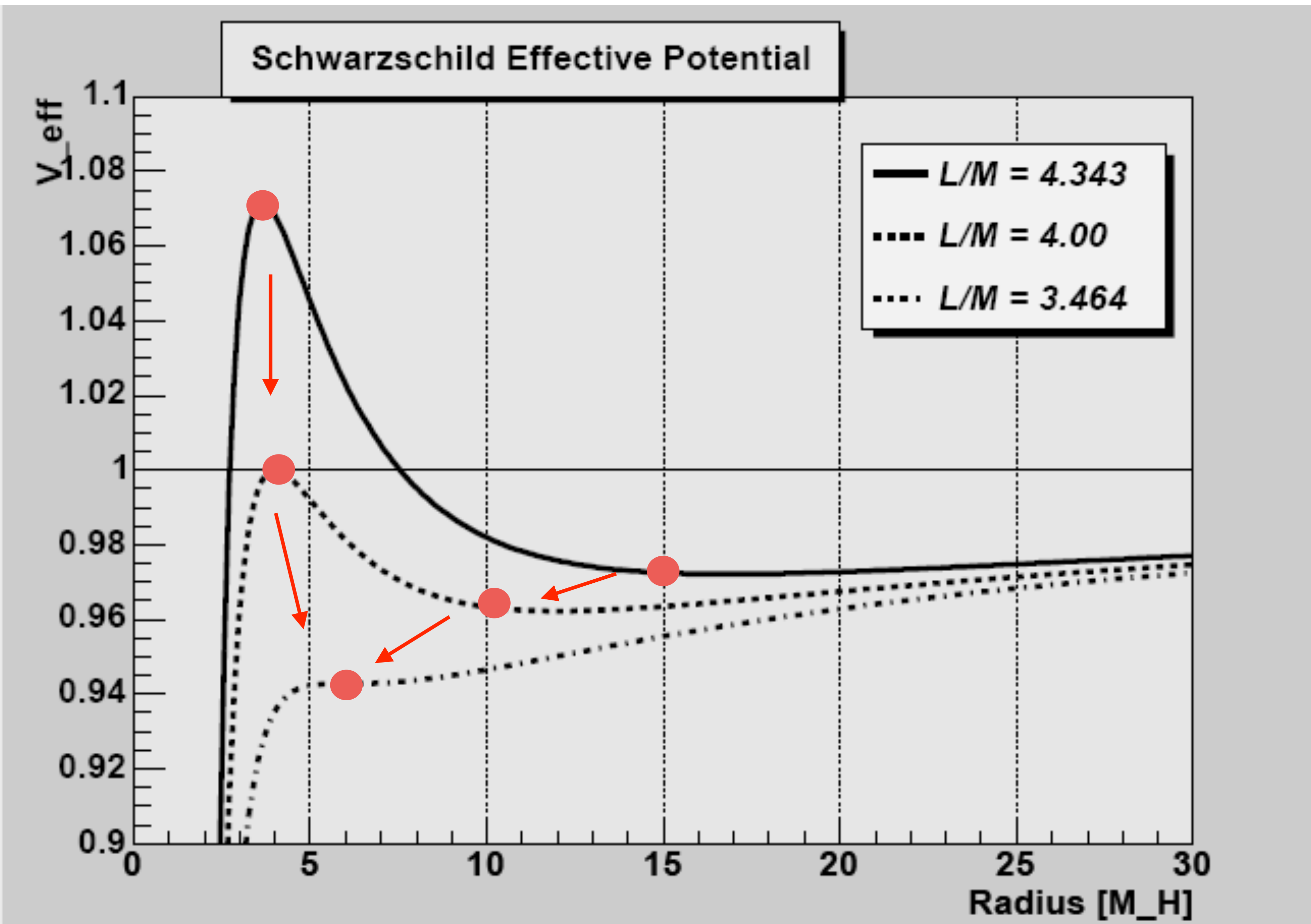
$$\frac{d\phi}{d\tau} = \frac{L}{r^2}$$

first equation allow us to define an effective potential

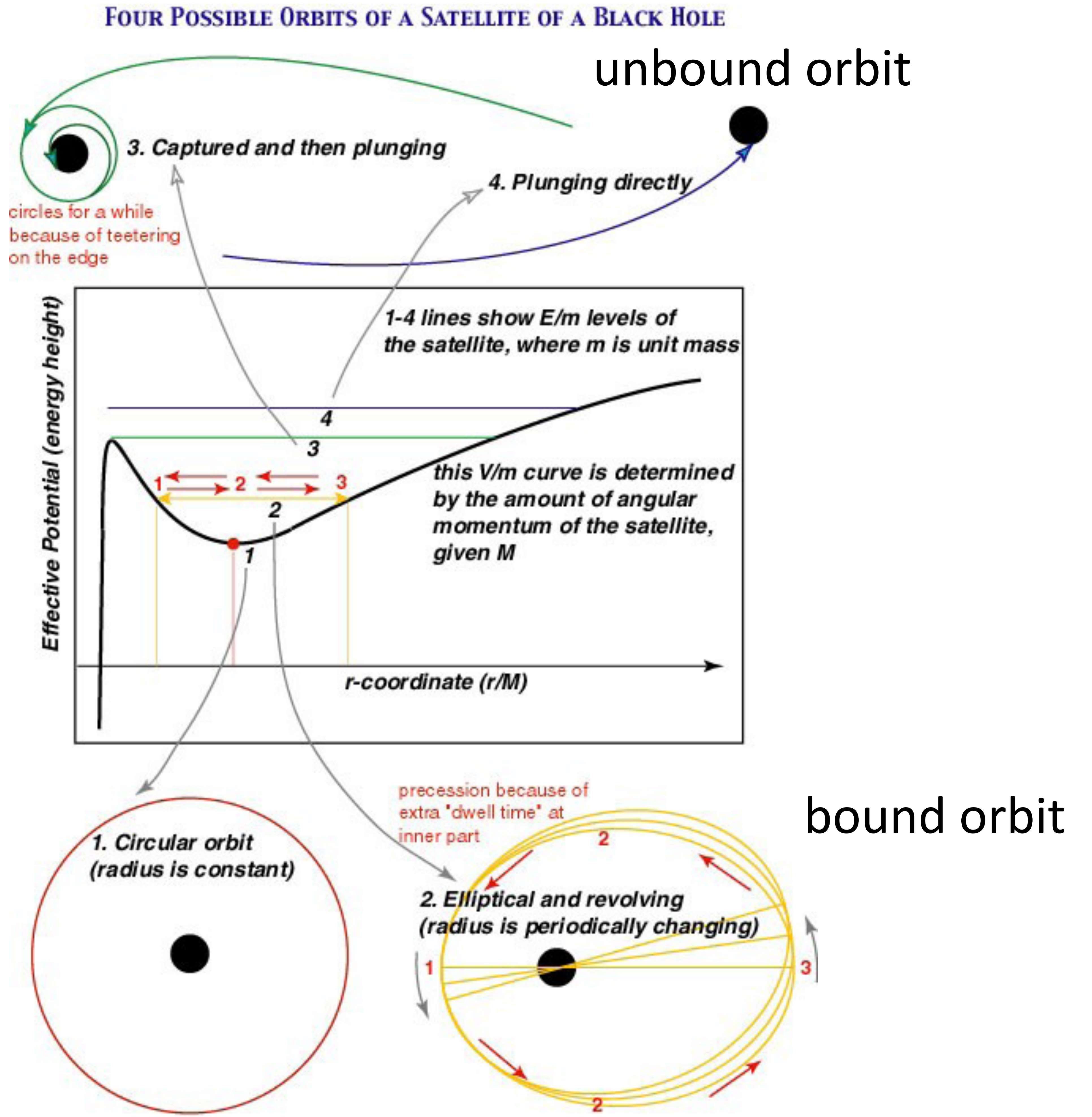
$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - V_{\rm eff}^2 \qquad V_{\rm eff} = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right)$$

Effective potential

$$V_{\text{eff}} = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right)$$



Possible Orbit



Circular Orbit

First derivative of the potential give us the circular orbit

$$\frac{dV_{\text{eff}}}{dr} = \frac{d}{dr}\left(\frac{M}{r} - \frac{L^2}{2r^2} + \frac{L^2 M}{r^3}\right) = -\frac{M}{r^2} + \frac{L^2}{r^3} - \frac{3L^2 M}{r^4}$$

$$Mr^2 - L^2 r + 3L^2 M = 0$$

Find the min and max of the radius (unstable & stable)

$$r_c = \frac{L^2 \pm \sqrt{L^4 - 12L^2M^2}}{2M} = \frac{L^2}{2M}\left(1 \pm \sqrt{1 - \frac{12M^2}{L^2}}\right)$$

The condition for the existence of a circular orbit: $L^2 > 12M^2$

For the minimum $L/M > \sqrt{12}$, smallest possible radius of stable circular orbit (ISCO) in Schwarzschild metric:

$$6M < r_c(\text{stable}) < +\infty$$

unstable circular orbits:

$$3M < r_c(\text{unstable}) < 6M$$

Kerr Black Hole

Kerr 1963:
Boyer-Lindquist: 1967

Boyer-Lindquist coordinates

$$ds^2 = -\alpha^2 dt^2 + \tilde{\omega}^2 (d\phi - \omega dt)^2 + (\rho^2/\Delta) dr^2 + \rho^2 d\theta^2$$

$\alpha = \sqrt{\frac{\rho^2 \Delta}{\Sigma^2}}$	lapse function
$\Delta = r^2 - 2Mr + a^2$	Horizon function
$\rho^2 = r^2 + a^2 \cos^2\theta$	generalized radius
$\Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta$	sigma potential
$\omega = \frac{2aMr}{\Sigma^2}$	frame-dragging frequency
$\tilde{\omega} = \frac{\Sigma}{\rho} \sin\theta$	cylindrical radius

Properties of Kerr solution

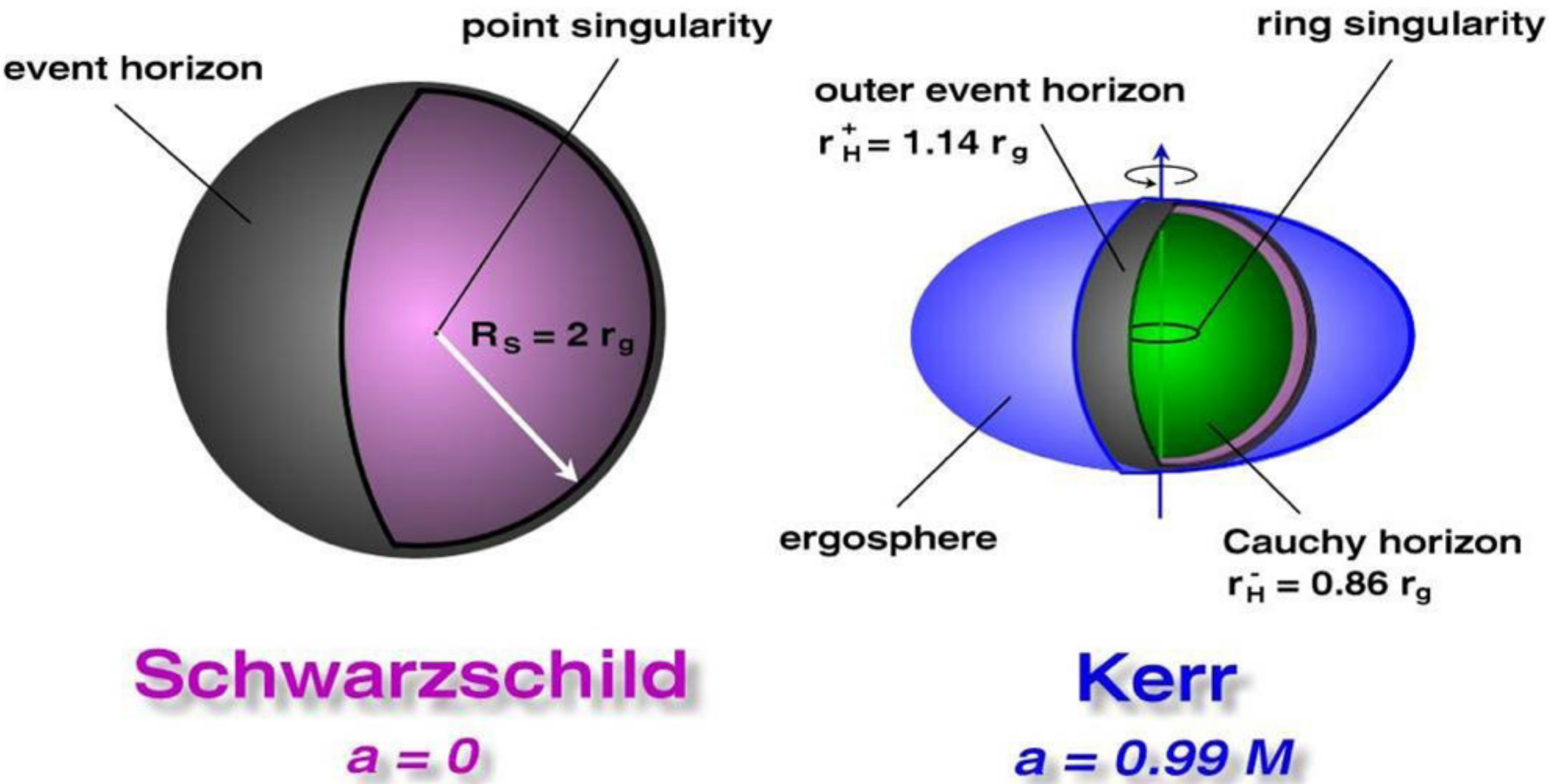
- Asymptotically flat: $\alpha^2 \sim 1 - 2M/r \quad \omega \sim 2J_H/r^3$

  $J_H = Ma$ angular momentum
- Event horizon: $\Delta(r_H) = 0$: $r_H = M + \sqrt{(M^2 - a^2)}$

  $a^2 < M^2$

Schwarzschild vs Kerr

Event horizon = surface of infinite redshift



Uniqueness of Kerr Solution

Robinson Theorem (1975)=> Black Holes have no hairs (Wheeler)

Stationary axisymmetric solutions of Einstein's vacuum equations which satisfy

(i) are asymptotically flat (Minkowskian)

(ii) contain a smooth convex horizon

(iii) are nonsingular outside the horizon

are uniquely specified by two parameters:
the mass M and the angular momentum

$$J_H = aM, J_H < M^2$$

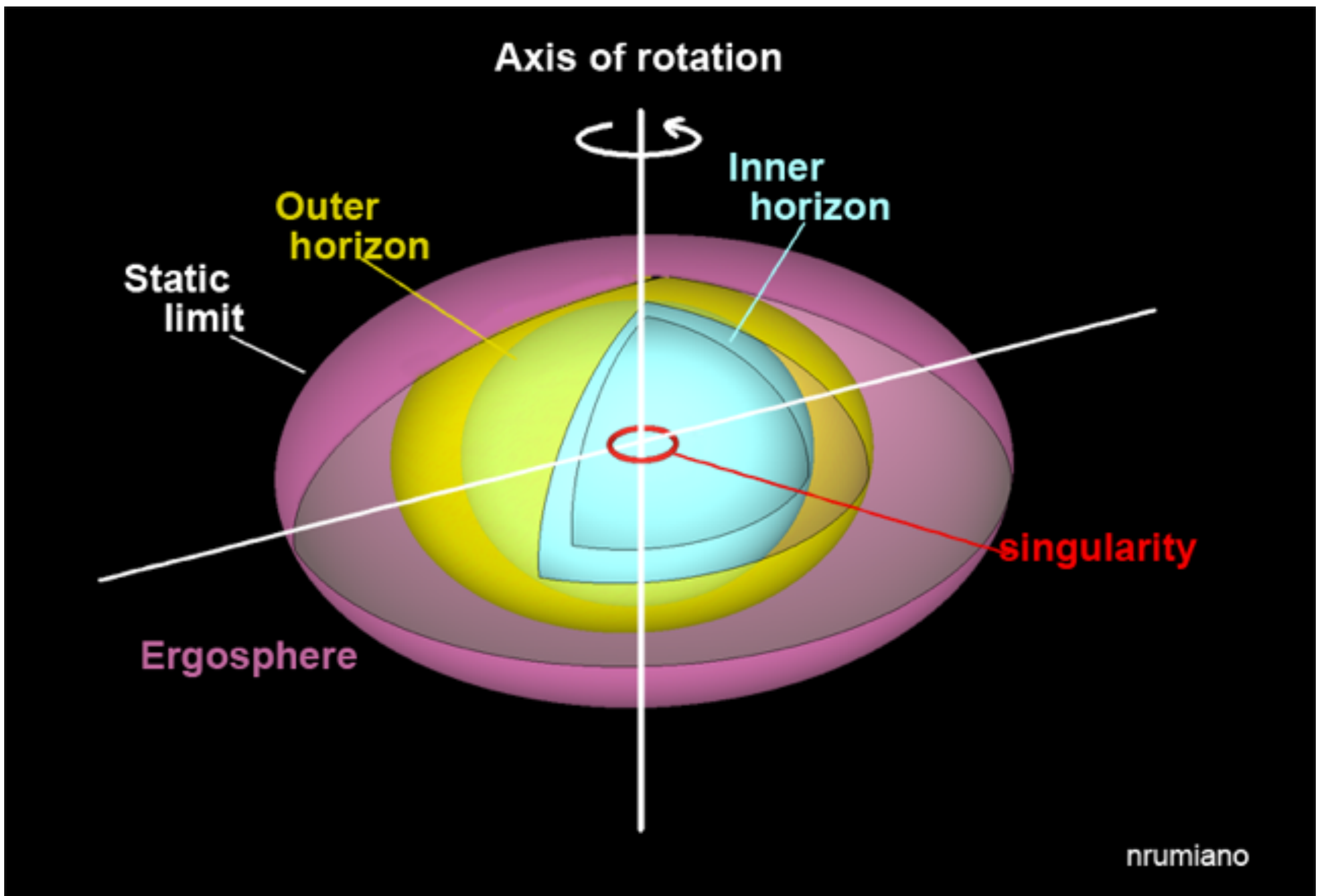
Stationary Black Hole Solution

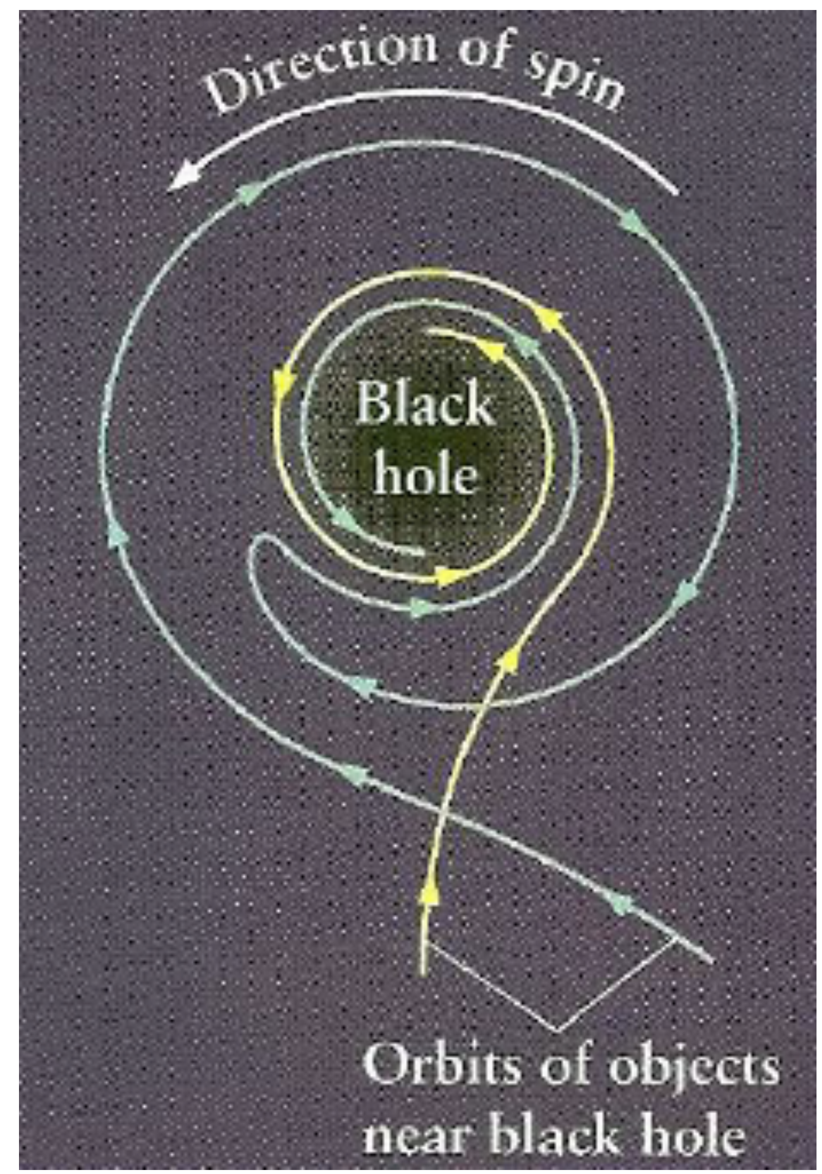
- mass
- angular momentum
- electric charge

- Schwarzschild
  {M}
- Reissner-Nordstrom
  {M, Q}
- Kerr
  {M, a}
- Kerr-Newman
  {M, a, Q}

Wheeler: no hair theorem

Ergosphere & Frame-dragging





Why “Ergoregion"?

- “Ergo” = Energy

- All the spin energy of a black hole resides outside the horizon => it can all be extracted (in … theory)

- For maximum rotating Kerr BH with mass M:

  spin energy = 29% of Mc^2

Two famous energy extraction mechanisms:

- Penrose process: particle splitting inside the elgosphere
- Blandford-Znajek process: BH spin twisted magnetic field

Geodesics in Equatorial Plane

Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(1-\frac{2Mr}{\rho^2}\right)\dot{t}^2 + \frac{2Mra\sin^2\theta}{\rho^2}\dot{t}\dot{\phi} - \frac{\rho^2}{2\Delta}\dot{r}^2$$
$$-\frac{\rho^2}{2}\dot{\theta}^2 - \frac{1}{2}\left[(r^2+a^2)\sin^2\theta + \frac{2Mra^2\sin^2\theta}{\rho^2}\right]\dot{\phi}^2$$

Conservation laws:

$$p_t = -E = \text{const.}$$
$$p_\phi = L = \text{const.}$$

$$E = -g_{t\mu}u^\mu = \left(1-\frac{2M}{r}\right)\dot{t} + \frac{2Ma}{r}\dot{\phi}$$

$$L = g_{\phi\mu}u^\mu = -\frac{2Ma}{r}\dot{t} + \left(r^2+a^2+\frac{2Ma^2}{r}\right)\dot{\phi}$$

=> radial equation

$$r^3\dot{r}^2 = E^2r^3 - r\Delta - r(L^2-a^2E^2) + 2M(aE-L)^2$$

Circular Orbits at Equator & ISCO in Kerr

radial equation

$$r^3\dot{r}^2 = E^2r^3 - r\Delta - r(L^2 - a^2E^2) + 2M(aE - L)^2$$

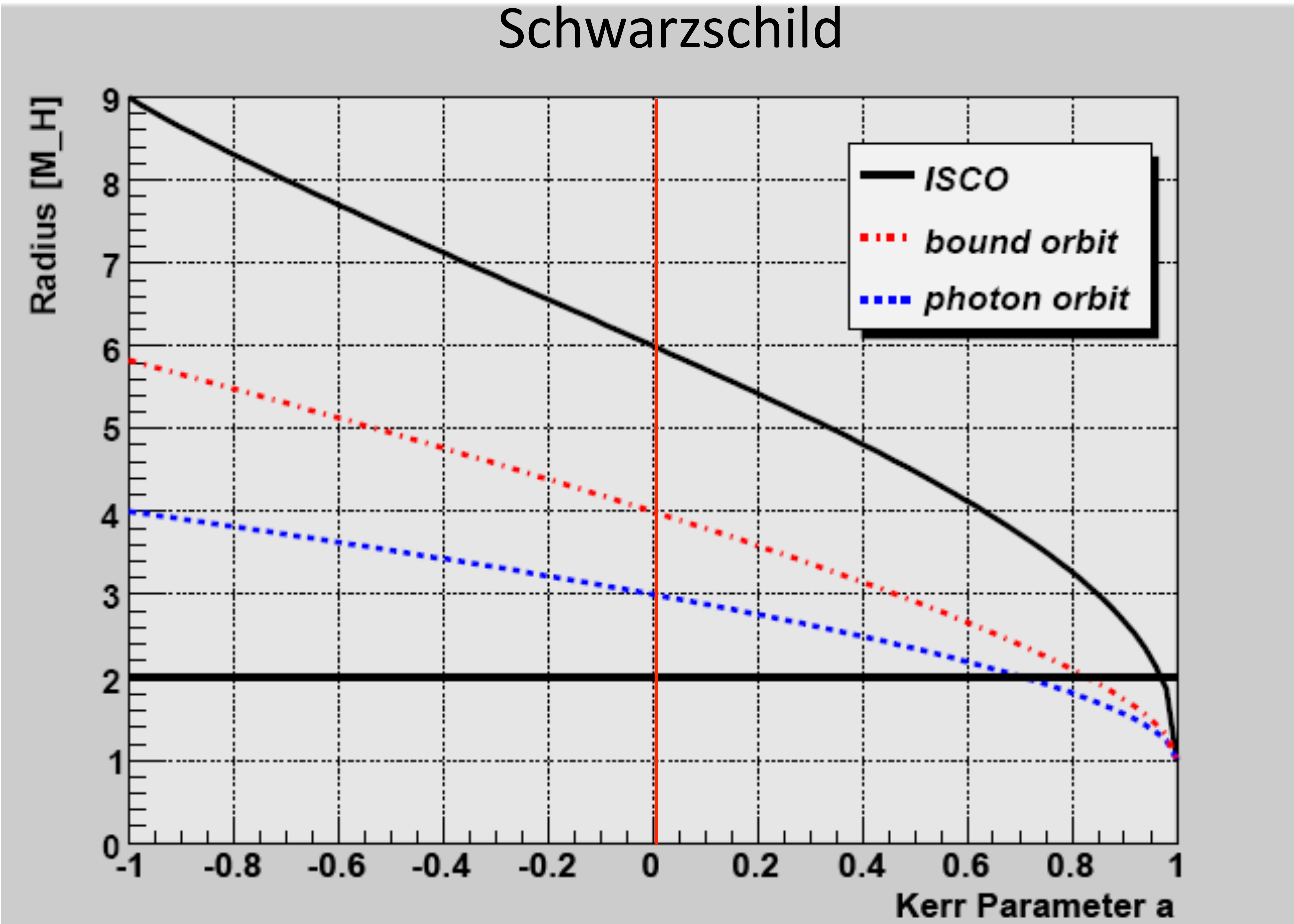
Looking for double roots

$$E = \frac{r^2 - 2Mr \mp a\sqrt{Mr}}{r\sqrt{r^2 - 3Mr \mp 2a\sqrt{Mr}}} \qquad L = \pm\frac{\sqrt{Mr}(r^2 - 2a\sqrt{Mr} + a^2)}{r\sqrt{r^2 - 3Mr \pm 2a\sqrt{Mr}}}$$

Looking for triple roots => ISCO radius

$$\begin{aligned} r_{ms} &= M(3 + Z_2 \mp \sqrt{(3 - Z_1)(3 + Z_1 + 2Z_2)}) \\ Z_1 &= 1 + \left(1 - \frac{a^2}{M^2}\right)^{1/3}\left(\left(1 + \frac{a}{M}\right)^{1/3} + \left(1 - \frac{a}{M}\right)^{1/3}\right) \\ Z_2 &= \sqrt{3\frac{a^2}{M^2} + Z_1^2} \end{aligned}$$

Characteristic Radii in Kerr



Summary - General Relativity

- Spacetime is the set of all events, it has the structure of a pseudo-Riemannian manifold with a metric tensor field g.

- Einstein‘s gravity assumes the connection to be metric

- Freely falling objects follow geodesics on this manifold

- The Einstein tensor is coupled to the energy-momentum tensor of all matter in the spacetime (including fields and vacuum).

- Black Hole is predicted from the GR (solution of Einstein’s field equation).

- From BH no hair theorem, BH has only three information (mass, angular momentum, & charges)