3+1 Form of General Relativistic MHD Equations

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General Relativistic Hydrodynamics

- The general relativistic hydrodynamics equations are obtained from the local conservation laws of the stress-energy tensor, $T_{\mu\nu}$ (the Bianchi identities), and of the matter current density J_μ (the continuity equation):

$$\nabla_\mu(\rho u^\mu) = 0 \qquad \nabla_\mu T^{\mu\nu} = 0$$

equations of motion $(\mu = 0, ..., 3)$

∇_μ : covariant derivative associated with the four dimensional spacetime metric $g_{\mu\nu}$

- The density current is given by $J^\mu = \rho u^\mu$

u^μ is the fluid 4-velocity and ρ is the rest-mass density in a locally inertial reference frame.

Energy-Momentum Tensor

The energy-momentum (stress-energy) tensor for a non-perfect fluid is defined as:

$$T^{\mu\nu} = \rho(1+\epsilon)u^\mu u^\nu + (p - \mu\Theta)h^{\mu\nu} - 2\xi\sigma^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu$$

where ε is the specific internal energy density of the fluid, p is the pressure, and $h^{\mu\nu}$ is the spatial projection tensor, $h^{\mu\nu}=u^\mu u^\nu+g^{\mu\nu}$.
In addition, μ and ξ are the shear and bulk viscosity coefficients.

The expansion, Θ, describing the divergence or convergence of the fluid world lines is defined as $\Theta=\nabla_\mu u^\nu$. The symmetric, tracefree, and spatial shear tensor $\sigma_{\mu\nu}$ is defined by:

$$\sigma^{\mu\nu} = \frac{1}{2}(\nabla_\alpha u^\mu h^{\alpha\nu} + \nabla_\alpha u^\nu h^{\alpha\mu}) - \frac{1}{3}\Theta h^{\mu\nu}$$

Finally q_μ is the energy flux vector.

Energy-Momentum Tensor

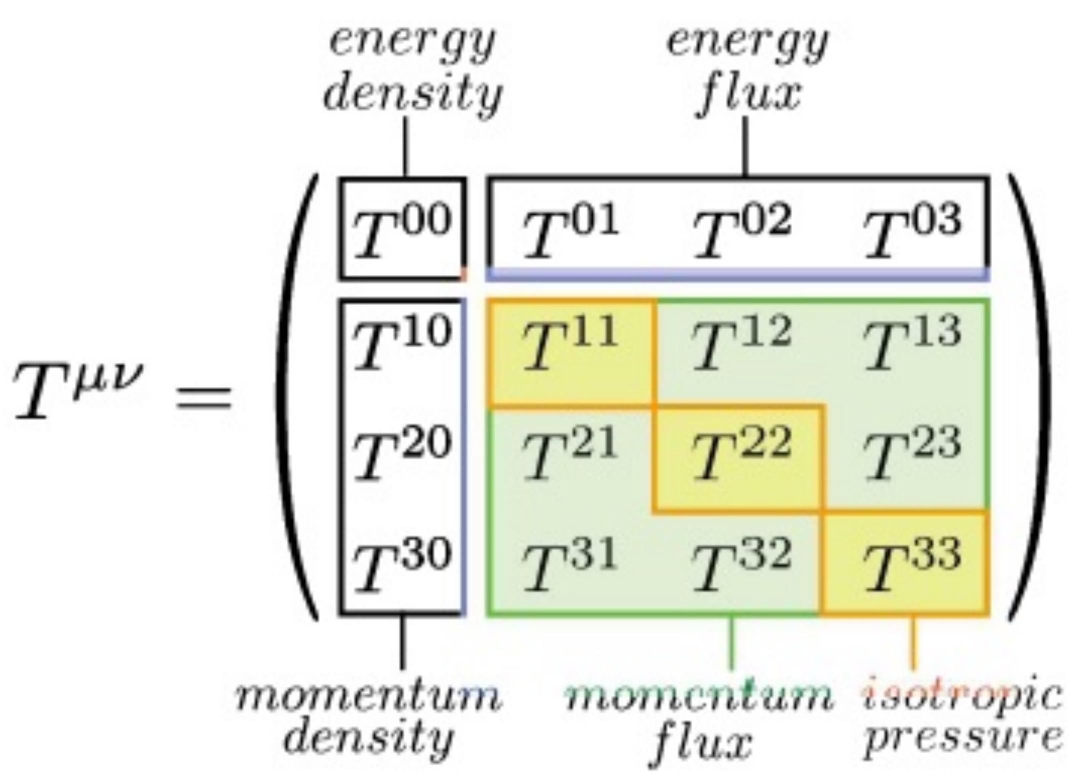
In the following we will neglect non-adiabatic effects, such as viscosity or heat transfer, the energy-momentum tensor is given that of a perfect fluid:

$$T^{\mu\nu} = \rho h u^\mu u^\nu + p g^{\mu\nu}$$

where we have introduced the relativistic specific enthalpy,

$$h = 1 + \epsilon + \frac{p}{\rho}$$

General form of energy-momentum tensor



General Relativistic Magnetohydrodynamics

- GRHD equations + Maxwell equations

$$\nabla_\mu(\rho u^\mu) = 0 \quad \nabla_\mu T^{\mu\nu} = 0 \quad \nabla_\mu {}^*F^{\mu\nu} = 0$$

where $F^{\mu\nu}$, Faraday tensor may be constructed from electric and magnetic fields E^α, B^α as measured in a generic frame $\mathcal{U}^\alpha$ as

$$F^{\mu\nu} = \mathcal{U}^\mu E^\nu - \mathcal{U}^\nu E^\mu - (-g)^{-1/2}\eta^{\mu\nu\lambda\delta}\mathcal{U}_\lambda B_\delta$$

where $\eta^{\mu\nu\lambda\delta}$: the fully-antisymmetric symbol and g : determinant of 4-metric

- The dual Faraday tensor is

$$^*F^{\mu\nu} = \mathcal{U}^\mu B^\nu - \mathcal{U}^\nu B^\mu - (-g)^{-1/2}\eta^{\mu\nu\lambda\delta}\mathcal{U}_\lambda E_\delta$$

- Ideal MHD limit

$$F^{\mu\nu}u_\nu = 0$$

$$J^\mu = \rho_q u^\mu + \sigma F^{\mu\nu}u_\nu \quad \sigma \to \infty$$

Energy-Momentum Tensor

- To eliminate the electric fields from the equations, it is convenient to introduce vectors in the fluid frame.
- The electric field and magnetic field 4-vectors:

$$e^{\mu} = F^{\mu\nu}u_{\nu}, \;\; b^{\mu} = {}^{*}F^{\mu\nu}u_{\nu} \quad \text{where } e^{\mu}{=}0 \text{ and } \; u_{\mu}b^{\mu} = 0$$

- The Faraday tensor is

$$\begin{aligned} F^{\mu\nu} &= -(-g)^{-1/2}\eta^{\mu\nu\lambda\delta}u_{\lambda}u_{\delta} \\ {}^{*}F^{\mu\nu} &= b^{\mu}u^{\nu} - b^{\nu}u^{\mu} \end{aligned}$$

- Energy-momentum tensor in ideal GRMHD is given by

$$T^{\mu\nu} = (\rho h + b^2)u^{\mu}u^{\nu} + \left(p + \frac{1}{2}b^2\right)g^{\mu\nu} - b^{\mu}b^{\nu}$$

where $b^2 = b^{\nu}b_{\nu}$ denoting the square of the fluid frame magnetic field strength as $b^2 = B^2 - E^2$

$$T^{\mu\nu} = T^{\mu\nu}_{\mathrm{fluid}} + T^{\mu\nu}_{\mathrm{EM}}$$

4-magnetic field

- Transformation between b^μ and B^μ

$$b^i = \frac{B^\mu + \alpha b^0 u^i}{W},\ b^0 = \frac{W(B^i v_i)}{\alpha}$$

- Faraday tensor can be expressed by B^μ provides evolution equation of magnetic field (induction equation)

$$^*F^{\mu\nu} = \frac{B^\mu u^\nu + B^\nu u^\mu}{W}$$

- The time component of Maxwell equation leads to the constraint of

$$\partial_i(\sqrt{\gamma} B^i) = 0$$

- The scalar b^2

$$b^2 = \frac{B^2 + \alpha^2 (b^0)^2}{W^2} = \frac{B^2}{W^2} + (B^i v_i)^2 \qquad B^2 = B^i B_i$$

Conservation laws

- Conservation laws with respect to an explicit coordinate chart $x^{\mu} = (x^0, x^i)$

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} (\sqrt{-g} \rho u^{\mu}) = 0$$
$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} (\sqrt{-g} T^{\mu\nu}) = \Gamma^{\nu}_{\mu\lambda} T^{\mu\lambda}$$
$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} (\sqrt{-g} \, {}^{*}F^{\mu\nu}) = 0$$

where the scalar x^0 represents a foliation of spacetime with hyper surfaces (with coordinates x^i) . where, $g=det(g_{\mu\nu})$, and $\Gamma^{\nu}_{\mu\lambda}$ are the Christoffel symbols.

Conservation laws

- The system formed by the eqs must be supplemented with an equation of state (EOS) relating the pressure to some fundamental thermodynamical quantities, e.g.

$$p = p(\rho, \epsilon)$$

ideal EoS: $p = (\Gamma - 1)\rho\epsilon$

polytropic EoS: $p = \kappa\rho^{\Gamma}, \ \Gamma = 1 + \frac{1}{N}$

- In the "test-fluid" approximation (fluid's self-gravity neglected), the dynamics of the matter fields is fully described by the previous conservation laws and the EOS.

- When such approximation does not hold, the previous equations must be solved in conjunction with Einstein's equations for the gravitational field which describe the evolution of a dynamical spacetime.

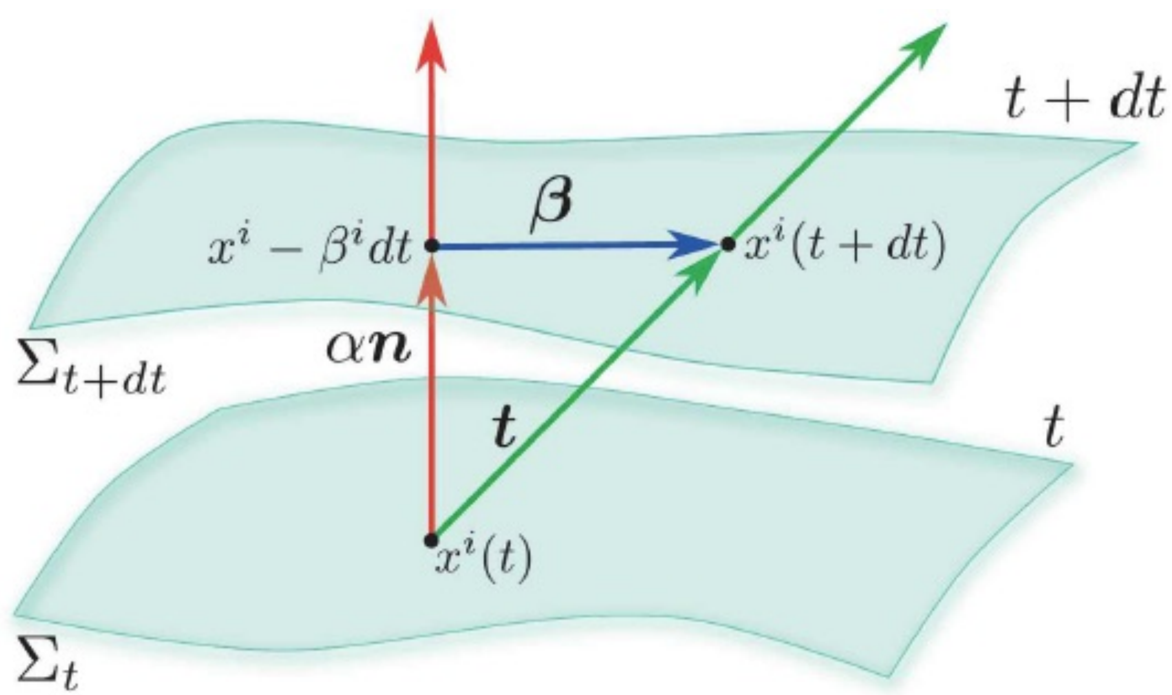
(Newtonian analogy: Euler's equation + Poisson's equation)

Solving GRMHD Eqs on Computer

- General relativity states that our world is a 4D and curved spacetime.
- The GRMHD equations describe its dynamics in 4D curved spacetime.
- How to solve the GRMHD equations numerically?
- Prominently, there is no a priori concept of "flowing of time", time is just one of the dimensions, and on the same level as space dimensions...
- There is a successful approach: 3+1 formulation.

Foliate the 4D Spacetime

- Spacetime is foliated with a set of non-intersecting space like hypersurfaces Σ. Within each surface distances are measured with the spatial 3-metric.



The time-like unit normal vector to the hyper surface: normalization condition

$$n_\mu = -\alpha \Omega_\mu = -\alpha \nabla_\mu t, \; (n^\mu n_\mu = -1)$$

α: lapse function, Ω_μ: the direction of time

$$n_\mu = (-\alpha, 0_i)$$

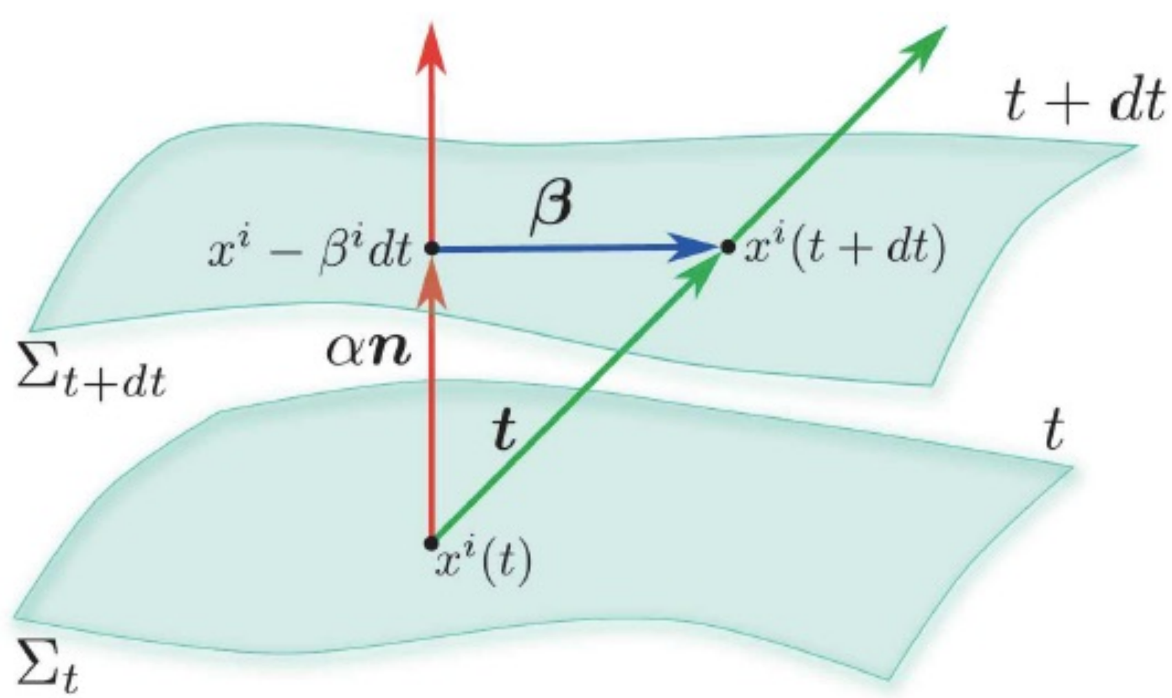
The observer moving with 4-velocity n^μ is called *Eulerian*

- Any vector V^μ may be projected in its temporal component: $V^{\hat{n}} = -n_\mu V^\mu$ and spatial component: $\perp V^\mu = (g^\mu_\nu + n^\mu n_\nu) V^\nu$
- 3D spatial metric associated to each hypersurface:

$$\gamma_{\mu\nu} = \perp g_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

Foliate the 4D Spacetime

- We must ensure that when going from one hypersurface Σ at time t to another Σ at time $t+dt$, all the vectors originating on Σ_t end up on Σ_{t+dt}.
- We must land on a single hypersurface.



- The most general of such vectors that connect two hypersurfaces is

$$t^\mu = \alpha n^\mu + \beta^\mu$$

where $\boldsymbol{\beta}$ is any spatial shift vector.

$$\beta^\mu = (0, \beta^i) \qquad \beta^\mu n_\mu = 0$$

unit vector components

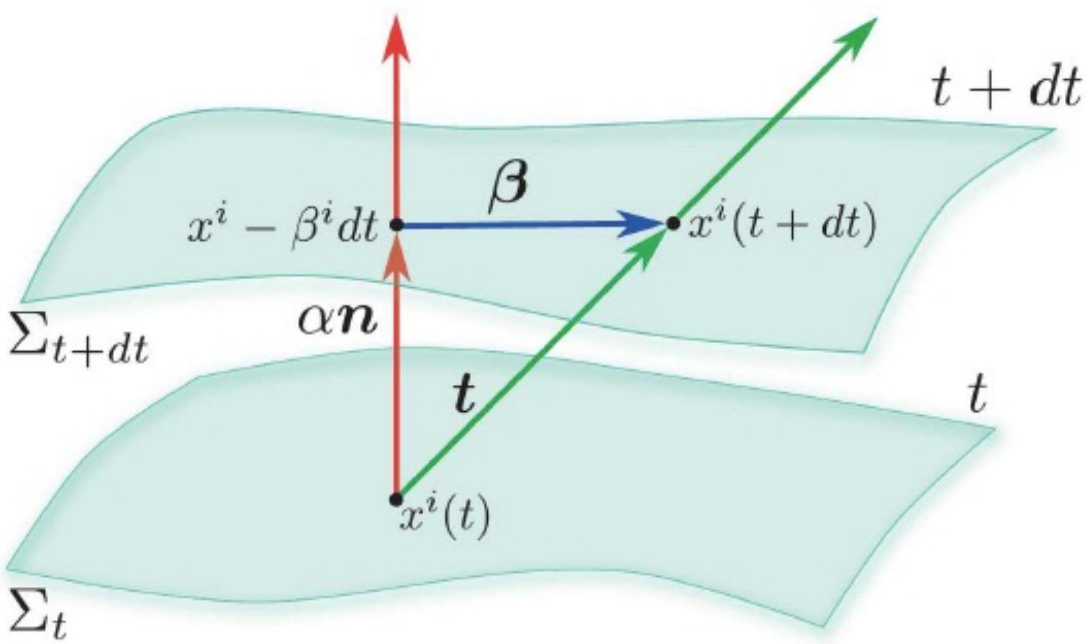
$$n_\mu = (-\alpha, 0_i), \;\; n^\mu = (1/\alpha, -\beta^i/\alpha)$$

Line element in 3+1 Form

$$x^\mu = (t, x^j)$$

- Generic line element in 3+1 decomposition:

$$ds^2 = -(\alpha^2 - \beta_i \beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j$$



- when $\beta^i = 0 = dx^i$ the lapse measures the proper time, dτ, between two adjacent hyper surfaces:

$$d\tau^2 = \alpha^2(t, x^j)dt^2$$

- shift vector measures the change of point from two hyper surfaces

$$x^i_{t+dt} = x^i_t - \beta^i(t, x^i)dt$$

- the spatial metric measures distances between points on hypersurface

$$dl^2 = \gamma_{ij} dx^i dx^j$$

Line element in 3+1 Form

- Covariant and contravariant metric can be written as

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_i \beta^i & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^i/\alpha \\ \beta^i/\alpha & \gamma^{ij} - \beta^i \beta^j/\alpha^2 \end{pmatrix}$$

$\sqrt{-g} = \alpha\sqrt{\gamma}$ where $g = \det(g_{\mu\nu}),\ \gamma = \det(\gamma_{ij})$

Spatial 4-velocity

- The spatial 4-velocity ***v*** measured by Eulerian observer is given by the ratio between the projection of 4-velocity ***u*** in the space orthogonal to ***n***, i.e., $\gamma^i_\mu u^\mu = u^i$ and Lorentz factor of ***u*** as measured by ***n***, $-n_\mu u^\mu = \alpha u^t$
- The spatial 4-velocity measured by Eulerian observer is

$$v^t = 0, \ v^i = \frac{\gamma^i_\mu u^\mu}{\alpha u^t} = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right)$$

$$v_t = \beta_i v^i, \ v_i = \frac{\gamma_{i\mu} u^\mu}{\alpha u^t} = \frac{u_i}{\alpha u^t} = \frac{\gamma_{ij}}{\alpha}\left(\frac{u^j}{u^t} + \beta^j\right)$$

- using $u^\mu u_\mu = -1$ $\qquad v^2 = v^i v_i$

$$\alpha u^t = -n_\mu u^\mu = \frac{1}{\sqrt{1 - v^i v_i}} = W \qquad u_t = W(-\alpha + \beta_i v^i)$$

- rewritten spatial 4-velocity

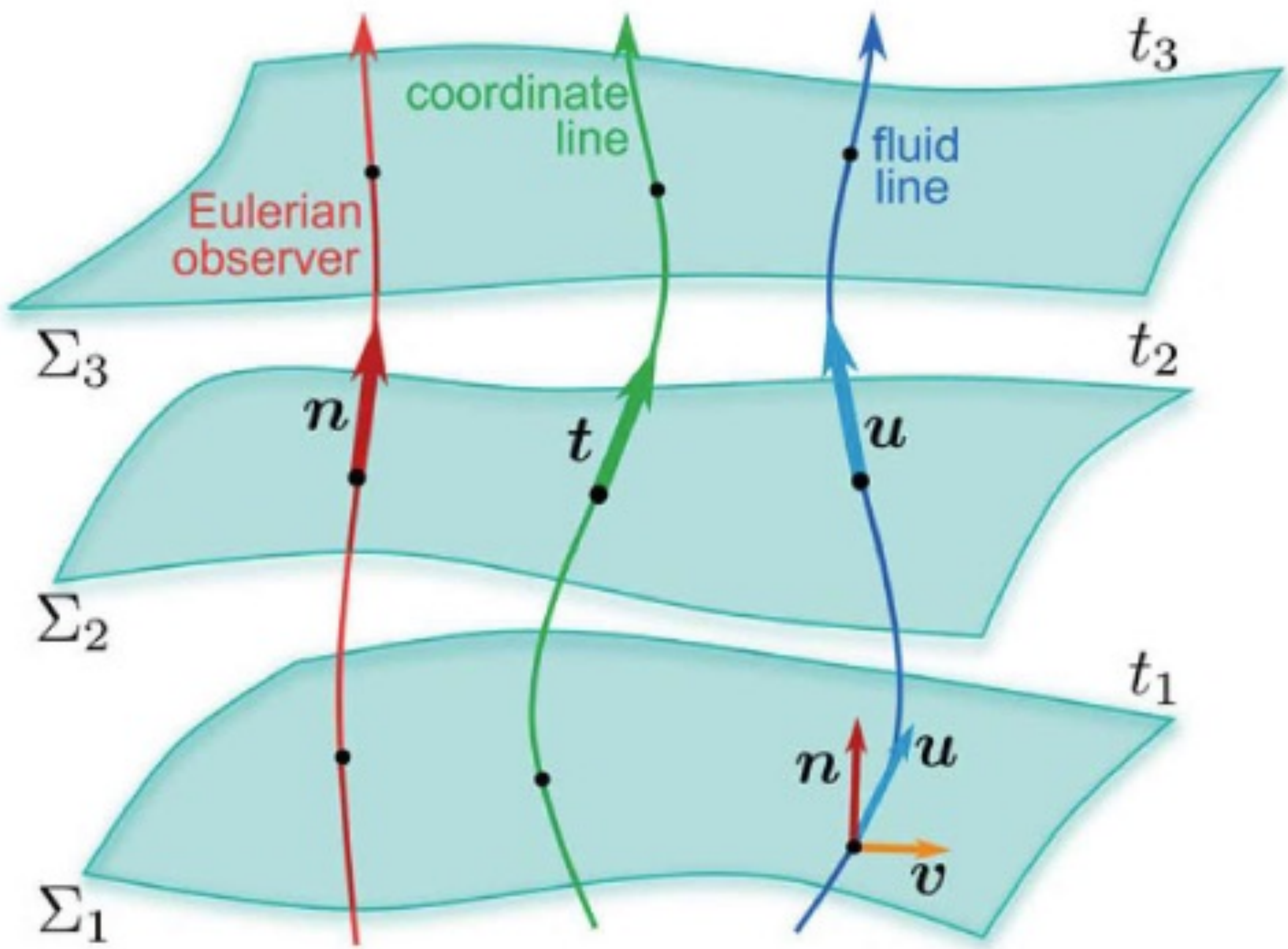
$$v^i = \frac{u^i}{W} + \frac{\beta^i}{\alpha} = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right) \qquad v_i = \frac{u_i}{W} = \frac{u_i}{\alpha u^t}$$

- transport velocity

$$u^i/u^t = \alpha v^i - \beta^i$$

Eulerian Observer

Eulerian observer: at rest on the hypersurface; moves from Σ_t to $\Sigma_{t+\Delta t}$ along the unit normal vector. Speed given by: $v^i = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right)$



3+1 Valencia formulation continuity equation

- Develop 3+1 formulation of general relativistic hydrodynamic equations.
- First consider continuity equation

$$\nabla_\mu(\rho u^\mu) = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\rho u^\mu)$$
$$= \frac{1}{\sqrt{-g}}[\partial_t(\sqrt{-g}\rho u^t) + \partial_i(\sqrt{-g}\rho u^i)] = 0$$

- conserved variable

$$D = -\rho u^\mu n_\mu = \rho\alpha u^t = \rho W$$

- rewrite the continuity equation

$$\boxed{\partial_t(\sqrt{\gamma}D) + \partial_i[\sqrt{\gamma}D(\alpha v^i - \beta^i)] = 0}$$

$$\partial_t(\sqrt{\gamma}D) + \partial_i[\sqrt{\gamma}D\mathcal{V}^i] = 0 \qquad \mathcal{V}^i = \alpha v^i - \beta^i$$

3+1 Valencia formulation energy-momentum tensor

- Write the energy-momentum tensor in terms of quantities measured by normal observer in 3+1 decomposition of spacetime.

$$\begin{aligned} T^{\mu\nu} &= \rho h u^\mu u^\nu + p g^{\mu\nu} \\ &= U n^\mu n^\nu + S^\mu n^\nu + S^\nu n^\mu + W^{\mu\nu} \end{aligned}$$

where

$U = n_\mu n_\nu T^{\mu\nu}$: conserved energy density as the component of T fully projected along the normal unit vector to the spatial hyper surface

$W^{ij} = \gamma^i_\mu \gamma^j_\nu T^{\mu\nu}$: spatial variant of energy-momentum tensor as the projection of T in the space orthogonal to n (pure spatial 4=>3)

$S^i = \gamma^i_\mu n_\nu T^{\mu\nu}$: contravariant 3-momentum density in the Eulerian frame as the mixed parallel-transverse component of T (spatial 4=> 3)

3+1 Valencia formulation energy-momentum tensor

- 4-velocity and metric in its 3+1 decomposed form

$$u^\mu = W(n^\mu + v^\mu) \qquad g^{\mu\nu} = \gamma^{\mu\nu} - n^\mu n^\nu$$

- In this way, we obtain

$$T^{\mu\nu} = \rho h W^2 (n^\mu + v^\mu)(n^\nu + n^\nu) + p(\gamma^{\mu\nu} - n^\mu n^\nu)$$

- After rearranging terms,

$$\begin{aligned} W^{\mu\nu} &= \rho h W^2 v^\mu v^\nu + p\gamma^{\mu\nu}, \\ S^\mu &= \rho h W^2 v^\mu, \\ U &= \rho h W^2 - p \end{aligned}$$

Here v^t=0,

$$\begin{aligned} W^{ij} &= \rho h W^2 v^i v^j + p\gamma^{ij}, \\ S^i &= \rho h W^2 v^i, \\ U &= \rho h W^2 - p \end{aligned}$$

3+1 Valencia formulation
energy & momentum equation

- next step write the 4-divergence of a symmetric rank-2 tensor:

$$\partial_\mu T^{\mu\nu} = g^{\mu\lambda}\left[\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}T^\mu_\lambda) - \frac{1}{2}T^{\alpha\beta}\partial_\lambda g_{\alpha\beta}\right]$$

- Applying energy-momentum tensor and using the conservation of energy and momentum: $\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}T^\mu_\nu) = \frac{1}{2}T^{\mu\lambda}\partial_\nu g_{\mu\lambda}$

- Applying decomposed form, the conversation of momentum equation:

$$\partial_t(\sqrt{\gamma}S_j) + \partial_i[\sqrt{\gamma}(\alpha W^i_j - \beta^i S_j)] = \frac{1}{2}\sqrt{-g}T^{\mu\nu}\partial_j g_{\mu\nu}$$

- The energy equation is obtained from

$$\nabla_\mu(T^{\mu\nu}n_\nu) - T^{\mu\nu}\nabla_\mu n_\nu = 0$$

- Replacing energy-momentum tensor in decomposed form

$$\partial_t(\sqrt{\gamma}U) + \partial_i[\sqrt{\gamma}(\alpha S^i - \beta^i U)] = -\sqrt{-g}T^{\mu\nu}\nabla_\mu n_\nu$$

3+1 Valencia formulation source term

- There are source term on right hand side in energy and momentum equations.
- Rewrite source term in momentum equation as

$$\frac{1}{2}\sqrt{-g}T^{\mu\nu}\partial_j g_{\mu\nu} = \sqrt{-g}\left(\frac{1}{2}W^{ik}\partial_j\gamma_{ik} + S^\mu n^\nu \partial_j g_{\mu\nu} + \frac{1}{2}Un^\mu n^\nu \partial_j g_{\mu\nu}\right)$$
$$= \sqrt{-g}\left(\frac{1}{2}W^{ik}\partial_j\gamma_{ik} + \frac{1}{\alpha}S_i\partial_j\beta^i - U\partial\ln\alpha\right)$$

where we used $\partial_j g_{\mu\nu} = \Gamma^\kappa_{j\nu}g_{\mu\kappa} + \Gamma^\kappa_{j\mu}g_{\kappa\nu}$

- Rewrite source term in energy equation as

$$-\sqrt{-g}T^{\mu\nu}\nabla_\mu n_\nu = \sqrt{-g}(K_{ij}W^{ij} - S^i\partial_i\ln\alpha)$$

where $K_{\mu\nu}$ is extrinsic curvature.

- If spacetime is stationary

$$\alpha W^{ik}K_{ik} = \frac{1}{2}W^{ik}\beta^j\partial_j\gamma_{ik} + W^j_i\partial_j\beta^i$$

3+1 Valencia formulation

hyperbolic system

$$\partial_t(\sqrt{\gamma}\boldsymbol{U}) + \partial_j(\sqrt{\gamma}\boldsymbol{F}^i) = \sqrt{\gamma}\boldsymbol{S}$$

conserved variables numerical flux

$$\mathcal{V}^i = \alpha v^i - \beta^i$$

$$\boldsymbol{U} = \begin{bmatrix} D \\ S_j \\ \tau \end{bmatrix};\ \boldsymbol{F}^i = \begin{bmatrix} \mathcal{V}^i D \\ \alpha W^i_j - \beta^i S_j \\ \alpha(S^i - v^i D) - \beta^i \tau \end{bmatrix}$$

$$\begin{aligned} D &= \rho W \\ \tau &= U - D \end{aligned}$$

source term

$$\boldsymbol{S} = \begin{bmatrix} 0 \\ \frac{1}{2}\alpha W^{ik}\partial_j\gamma_{ik} + S_i\partial_j\beta^i - U\partial_j\alpha \\ \frac{1}{2}W^{ik}\beta^j\partial_j\gamma_{ik} + W^j_i\partial_j\beta^i - S^j\partial_j\alpha \end{bmatrix}$$

$$\begin{aligned} W^{ij} &= \rho h W^2 v^i v^j + p\gamma^{ij}, \\ S^i &= \rho h W^2 v^i, \\ U &= \rho h W^2 - p \end{aligned}$$

3+1 Valencia formulation

hyperbolic system

$$\partial_t(\sqrt{\gamma}\boldsymbol{U}) + \partial_j(\sqrt{-g}\boldsymbol{F}^i) = \sqrt{-g}\boldsymbol{S}$$

Yet another 3+1 Valencia form

conserved variables numerical flux

$$\boldsymbol{U} = \begin{bmatrix} D \\ S_j \\ \tau \end{bmatrix}; \ \boldsymbol{F}^i = \begin{bmatrix} D\tilde{v}^i \\ S_j\tilde{v}^i + p\delta^i_j \\ \tau\tilde{v}^i + pv^i \end{bmatrix}$$

$$\tilde{v}^i = v^i - \beta^i/\alpha$$

source term

$$\boldsymbol{S} = \begin{bmatrix} 0 \\ T^{\mu\nu}(\partial_\mu g_{\nu j} - \Gamma^\delta_{\nu\mu} g_{\delta j}) \\ \alpha(T^{\mu 0}\partial_\mu \ln\alpha - T^{\mu\nu}\Gamma^0_{\nu\mu}) \end{bmatrix}$$

$$\begin{aligned} D &= \rho W \\ S_j &= \rho h W^2 v_j \\ \tau &= \rho h W^2 - p - D \end{aligned}$$

Recovering special relativity & Newtonian limits

Full GR
(shift vector = 0)

$$\partial_t(\sqrt{\gamma}\rho W) + \partial_i(\sqrt{-g}\rho W v^i) = 0$$
$$\partial_t(\sqrt{\gamma}\rho h W^2 v_i) + \partial_i[\sqrt{-g}(\rho W^2 v^i v_j + p\gamma^i_j)] = (1/2)\sqrt{-g}T^{\mu\nu}\partial_j g_{\mu\nu}$$
$$\partial_t[\sqrt{\gamma}(\rho h W^2 - p)] + \partial_i(\sqrt{-g}\rho h W^2 v^i) = -\sqrt{-g}T^{\mu\nu}\nabla_\mu n_\nu$$

Minkowski
(covariant and contravariant is same)

$$\partial_t(\rho W) + \partial_i(\rho W v^i) = 0$$
$$\partial_t(\rho h W^2 v^i) + \partial_i(\rho W^2 v^i v^j + p\delta^{ij}) = 0$$
$$\partial_t(\rho h W^2 - p) + \partial_i(\rho h W^2 v^i) = 0$$

Newtonian
(remove relativistic correction)

$$\partial_t(\rho) + \partial_i(\rho v^i) = 0$$
$$\partial_t(\rho v^i) + \partial_i(\rho v^i v^j + p\delta^{ij}) = 0$$
$$\partial_t(\rho\epsilon + \tfrac{1}{2}\rho v^2) + \partial_i[(\rho\epsilon + \tfrac{1}{2}\rho v^2 + p)v^i] = 0$$

Eigenvalues (characteristic speeds)

- Numerical schemes based on Riemann solvers use the local characteristic structure of the hyperbolic system of equations.

- The eigenvalues (characteristic speeds) are all real (but not distinct, one showing a threefold degeneracy), and a complete set of right-eigenvectors exists. The above system satisfies, hence, the definition of hyperbolicity

Jacobian $$\boldsymbol{A}^{(i)} = \frac{\partial(\sqrt{\gamma}\boldsymbol{F}^i)}{(\sqrt{\gamma}\boldsymbol{U})} = \frac{\partial \boldsymbol{F}^i}{\partial \boldsymbol{U}}$$

Eigenvalues (along the x direction) $\boldsymbol{A}^{(x)}$

$$\lambda'_0 = v^x \quad \text{(triple)}$$

$$\lambda'_\pm = \frac{1}{1-v^2c_s^2}\left\{v^x(1-c_s^2) \pm c_s\sqrt{(1-v^2)[\gamma^{xx}(1-v^2c_s^2)-v^xv^x(1-c_s^2)]}\right\}$$

$$\lambda^i = \alpha\lambda'^i - \beta^i \quad \text{(GR correction)}$$

Eigenvalues (characteristic speeds)

Special relativistic limit along x-direction

$$\begin{aligned}\lambda'_0 &= v^x \\ \lambda'_\pm &= \frac{1}{1 - v^2 c_s^2}\left\{ v^x(1-c_s^2) \pm c_s\sqrt{(1-v^2)\left[\gamma^{xx}(1-v^2c_s^2) - v^x v^x(1-c_s^2)\right]}\right\}\end{aligned}$$

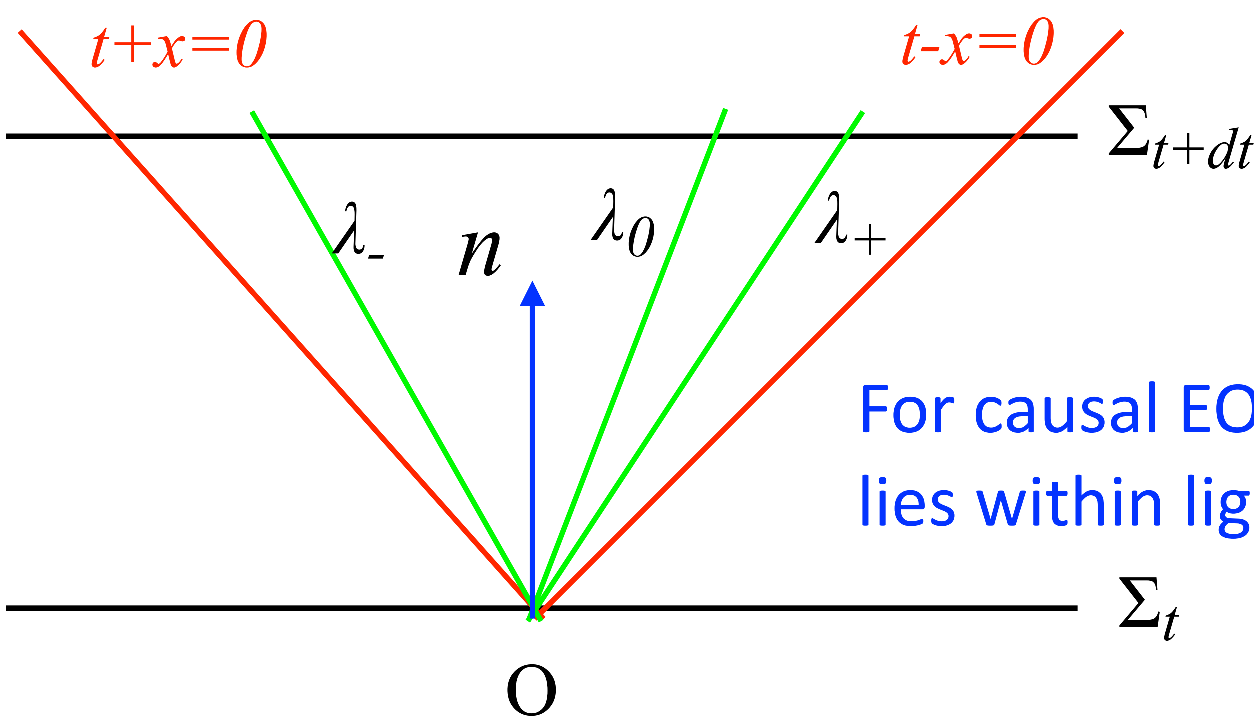
coupling with transversal components of the velocity (important difference with Newtonian case)

Even in pure 1D case:

$$\boldsymbol{v} = (v^x, 0, 0) \Rightarrow \lambda_0 = v^x,\ \ \lambda_\pm = \frac{v^x \pm c_s}{1 \pm v^x c_s}$$

Recall Newtonian (1D) case:

$$\lambda_0 = v^x,\ \ \lambda_\pm = v^x \pm c_s$$



For causal EOS sound cone lies within light cone

3+1 form of GRMHD equations

3+1 Decomposed energy-momentum tensor in GRMHD

conserved energy density

$$
\begin{aligned}
\mathcal{U} &= T^{\mu\nu} n_\mu n_\nu \\
&= \rho h W^2 - p + \frac{1}{2}(E^2 + B^2) \\
&= \rho h W^2 - p + \frac{1}{2}[B^2(1 + v^2) - (B^j v_j)^2]
\end{aligned}
$$

3-momentum density

$$
\begin{aligned}
S_i &= \gamma_i^\mu n^\nu T_{\mu\nu} = \rho h W^2 v_i + \eta_{ijk}\sqrt{\gamma} E^j B^k \\
&= \rho h W^2 v_i + B^2 v_i - (B^j v_j) B_i
\end{aligned}
$$

spatial variant of energy-momentum tensor

$$
\begin{aligned}
W^{ij} &= \gamma_\mu^i \gamma_\nu^j T_{\mu\nu} \\
&= \rho W^2 v^i v^j - E^i E^j + B^i B^j + \left[p + \frac{1}{2}(E^2 + B^2)\right]\gamma^{ij} \\
&= S^i v^j + p_{tot}\gamma^{ij} - (B^i B^j)/W^2 - (B_k v^k) v^i B^j
\end{aligned}
$$

3+1 form of GRMHD equations

hyperbolic system

$$\partial_t(\sqrt{\gamma}\boldsymbol{U}) + \partial_j(\sqrt{\gamma}\boldsymbol{F}^i) = \sqrt{\gamma}\boldsymbol{S}$$

conserved variables numerical flux

$$\boldsymbol{U} = \left[\begin{array}{c} D \\ S_j \\ \tau \\ B^j \end{array}\right] ; \ \boldsymbol{F}^i = \left[\begin{array}{c} \mathcal{V}^i D \\ \alpha W^i_j - \beta^i S_j \\ \alpha(S^i - v^i D) - \beta^i \tau \\ \mathcal{V}^i B^j - B^i \mathcal{V}^j \end{array}\right]$$

$$\mathcal{V}^i = \alpha v^i - \beta^i$$

source term

$$\boldsymbol{S} = \left[\begin{array}{c} 0 \\ \frac{1}{2}\alpha W^{ik}\partial_j\gamma_{ik} + S_i\partial_j\beta^i - U\partial_j\alpha \\ \frac{1}{2}W^{ik}\beta^j\partial_j\gamma_{ik} + W^j_i\partial_j\beta^i - S^j\partial_j\alpha \\ 0^j \end{array}\right]$$

$$\begin{aligned} W^{ij} &= S^i v^j + p_{tot}\gamma^{ij} - (B^i B^j)/W^2 - (B_k v^k)v^i B^j \\ S_i &= \rho h W^2 v_i + B^2 v_i - (B^j v_j)B_i \\ \mathcal{U} &= \rho h W^2 - p + \frac{1}{2}[B^2(1+v^2) - (B^j v_j)^2] \end{aligned}$$

3+1 form of GRMHD equations

hyperbolic system

Yet another 3+1 Valencia form

$$\partial_t(\sqrt{\gamma}\boldsymbol{U}) + \partial_j(\sqrt{-g}\boldsymbol{F}^i) = \sqrt{-g}\boldsymbol{S}$$

conserved variables numerical flux

$$\boldsymbol{U} = \left[\begin{array}{c} D \\ S_j \\ \tau \\ B^j \end{array}\right]; \ \boldsymbol{F}^i = \left[\begin{array}{c} D\tilde{v}^i \\ S_j\tilde{v}^i + p^*\delta^i_j - b_j B^i/W \\ \tau\tilde{v}^i + p^* v^i - \alpha b^0 B^i/W \\ \tilde{v}^i B^j - \tilde{v}^j B^i \end{array}\right] \qquad \tilde{v}^i = v^i - \beta^i/\alpha$$

source term

$$\boldsymbol{S} = \left[\begin{array}{c} 0 \\ T^{\mu\nu}(\partial_\mu g_{\nu j} - \Gamma^\delta_{\nu\mu} g_{\delta j}) \\ \alpha(T^{\mu 0})\partial_\mu \ln\alpha - T^{\mu\nu}\Gamma^0_{\nu\mu} \\ 0^j \end{array}\right]$$

$$\begin{aligned} D &= \rho W \\ S_j &= \rho h^* W^2 v_j - \alpha b_j b^0 \\ \tau &= \rho h^* W^2 - p - \alpha^2 (b^0)^2 - D \end{aligned}$$

$$b^i = \frac{B^\mu + \alpha b^0 u^i}{W}, \ b^0 = \frac{W(B^i v_i)}{\alpha}$$

3+1 form of GLM-GRMHD equations

hyperbolic system

$$\partial_t(\sqrt{\gamma}\boldsymbol{U}) + \partial_j(\sqrt{\gamma}\boldsymbol{F}^i) = \sqrt{\gamma}\boldsymbol{S}$$

conserved variables numerical flux

$$\boldsymbol{U} = \begin{bmatrix} D \\ S_j \\ \tau \\ B^j \\ \phi \end{bmatrix} ; \ \boldsymbol{F}^i = \begin{bmatrix} \mathcal{V}^i D \\ \alpha W^i_j - \beta^i S_j \\ \alpha(S^i - v^i D) - \beta^i \tau \\ \mathcal{V}^i B^j - \mathcal{V}^j B^i - B^i \beta^j \\ \alpha B^i - \phi\beta^i \end{bmatrix}$$

$$\mathcal{V}^i = \alpha v^i - \beta^i$$

- For magnetic monopole-control on AMR-grids, we solve the "Augented Faraday's law"

$$\nabla_\nu(F^{*\mu\nu} - \phi g^{\mu\nu}) = -\kappa n^\mu \phi$$

it is called Dedner cleaning, constraint dampening or GLM approach

source term

$$\boldsymbol{S} = \begin{bmatrix} 0 \\ \frac{1}{2}\alpha W^{ik}\partial_j\gamma_{ik} + S_i\partial_j\beta^i - \mathcal{U}\partial_j\alpha \\ \frac{1}{2}W^{ik}\beta^j\partial_i\gamma_{ik} + W^j_i\partial_j\beta^i - S^j\partial_j\alpha \\ -B^i\partial_i\beta^j - \alpha\gamma^{ij}\partial_i\phi \\ -\alpha\kappa\phi - \phi\partial_i\beta^i - \frac{1}{2}\phi\gamma^{ij}\beta^k\partial_k\gamma_{ij} + B^i\partial_i\alpha \end{bmatrix}$$

Eigenvalues in MHD

- Wave structure Newtonian MHD (Brio & Wu 1988): 7 physical waves

Two Alfven waves: $\lambda_{a_\pm} \Rightarrow \lambda_a = v_x \pm v_a$

Two Fast magnetosonic waves: $\lambda_{f_\pm} \Rightarrow \lambda_{f_\pm} = v_x \pm v_f$

Two Slow magnetosonic waves: $\lambda_{s_\pm} \Rightarrow \lambda_{s_\pm} = v_x \pm v_s$

One Entropy wave: $\lambda_e \Rightarrow \lambda_e = v_x$ $\qquad \lambda_{f_-} < \lambda_{a_-} < \lambda_{s_-} < \lambda_e < \lambda_{s_+} < \lambda_{a_+} < \lambda_{f_+}$

$$v_{f,s}^2 = \frac{1}{2}\left\{c_s^2 + \frac{B_x^2 + B_y^2 + B_z^2}{\rho} \pm \sqrt{\left(c_s^2 + \frac{B_x^2 + B_y^2 + B_z^2}{\rho}\right)^2 - 4\left(\frac{B_x^2}{\rho}\right) c_s^2}\right\}, \quad v_a = \sqrt{\frac{B_x^2}{\rho}}$$

- Wave structure for relativistic MHD (Anile 1989): roots of the characteristic equation.

- Only entropic waves and Alfvén waves are explicit.
- Magnetosonic waves are given by the numerical solution of a quartic equation.

Eigenvalues in GRMHD

- Simplified dispersion relation

$$w^2 = a^2 k^2$$

$$a^2 = c_s^2 + v_a^2 - c_s^2 v_a^2$$

$$c_s^2 = \frac{\Gamma p}{\rho h},\ \ v_a^2 = \frac{b^2}{\rho h + b^2}$$

- But simplified dispersion relation overestimate the wave speed in the fluid frame by up to a factor of 2, yielding a slightly more diffusive behavior

- Another simplified isotropic dispersion relation (similar to SRMHD case)

$$\lambda'^{i}_{\pm} = ((1-a^2)v^i \pm \sqrt{a^2(1-v^2)[(1-v^2a^2)\gamma^{ii} - (1-a^2)(v^i)^2]})/(1-v^2a^2)$$

$\lambda^i = \alpha\lambda'^i - \beta^i$ (GR correction)

Numerical Simulation Tips

- 3+1 form of GRMHD equations are set of hyperbolic equations. We can apply HRSC scheme.

- Difference from Newtonian MHD is that the calculation of primitive variables from conserved variables is not straightforward, need numerical calculation such as Newton-Raphson method.

C to P inversion procedure

- The GRMHD code require a calculation of primitive variables from conservative variables.
- The forward transformation (primitive → conserved) has a close-form solution, but the inverse transformation (conserved → primitive) requires the solution of a set of five nonlinear equations

$$
\begin{aligned}
D &= \rho W \\
S_j &= \rho h^* W^2 v_j - \alpha b_j b^0 \\
\tau &= \rho h^* W^2 - p - \alpha^2 (b^0)^2 - D
\end{aligned}
$$

$$
b^i = \frac{B^\mu + \alpha b^0 u^i}{W}, \quad b^0 = \frac{W(B^i v_i)}{\alpha}
$$

Method

- Noble's 2D method (Noble et al. 2005)
- Mignone & McKinney's method (Mignone & McKinney 2007)
- etc

Need numerical method (e.g., Newton-Raphson) to solving nonlinear equations

Horizon-penetrating form of BH metric

- Boyer-Lindquist coordinates of Black Hole metric has coordinate singularity at black hole event horizon radius.
- The metric term is diverge. We can not solve numerically at event horizon. => Should set inner boundary outside BH event horizon.
- But we can choose particular observer, such coordinate singularity can remove.
- Such BH metric form is so-called horizon-penetrating form
- In Schwarzschild BH: Eddington-Finkelstein coordinates, Lemaitre coordinates. In Kerr BH: Kerr-Schild coordinates

Kerr-Schild BH metric

line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

in spherical coordinate:

$$g_{tt} = -\left(1 - \frac{2Mr}{\rho^2}\right), \; g_{tr} = g_{rt} = \frac{2Mr}{\rho^2}, \; g_{t\phi} = g_{\phi t} = -\frac{2Mar\sin^2\theta}{\rho^2}$$

$$g_{rr} = 1 + \frac{2Mr}{\rho^2}, \; g_{r\phi} = g_{\phi r} = -a\sin^2\theta\left(1 + \frac{2Mr}{\rho^2}\right), \; g_{\theta\theta} = \rho^2, \; g_{\phi\phi} = \frac{A\sin^2\theta}{\rho^2}$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \; \Delta = r^2 - 2Mr + a^2, \; A = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$$

Note: original Kerr-Schild coordinate is written in Cartesian coordinate

Kerr-Schild BH metric

line element (3+1 decomposed form)

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_i \beta^i & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^i/\alpha \\ \beta^i/\alpha & \gamma^{ij} - \beta^i \beta^j/\alpha^2 \end{pmatrix}$$

$$\beta_i = \gamma_{ij}\beta^j$$

$$\alpha = \left(1 + \frac{2Mr}{\rho^2}\right)^{-1/2}, \; \beta^r = \frac{2Mr/\rho^2}{1 + 2Mr/\rho^2}$$

$$\gamma_{rr} = 1 + \frac{2Mr}{\rho^2}, \; \gamma_{\theta\theta} = \rho^2, \; \gamma_{\phi\phi} = \frac{A^2 \sin^2\theta}{\rho^2},$$

$$\gamma_{r\phi} = \gamma_{\phi r} = -a \sin^2\theta \left(1 + \frac{2Mr}{\rho^2}\right)$$

$$\rho^2 = r^2 + a^2 \cos^2\theta, \; \Delta = r^2 - 2Mr + a^2, \; A = (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta$$

Kerr BH metric

For the comparison with Kerr-Shild metric

line element (3+1 decomposed form)

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_i \beta^i & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^i/\alpha \\ \beta^i/\alpha & \gamma^{ij} - \beta^i \beta^j/\alpha^2 \end{pmatrix}$$

$$\beta_i = \gamma_{ij}\beta^j$$

$$\alpha = \sqrt{\frac{\rho^2 \Delta}{A}},\ \beta^\phi = -\omega = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2Mar}{A}$$

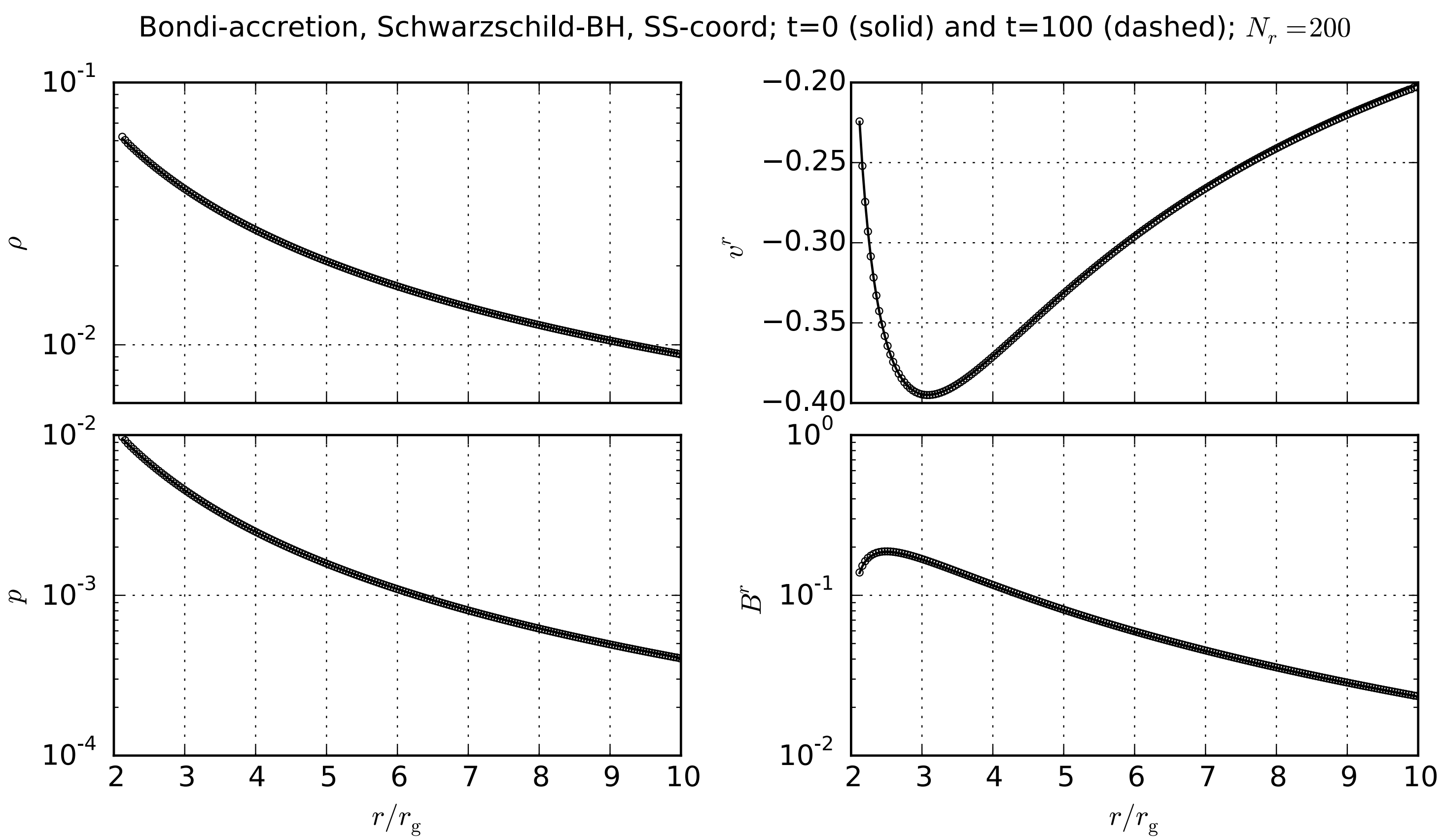
$$\gamma_{rr} = \frac{\rho^2}{\Delta},\ \gamma_{\theta\theta} = \rho^2,\ \gamma_{\phi\phi} = \frac{A \sin^2\theta}{\rho^2}$$

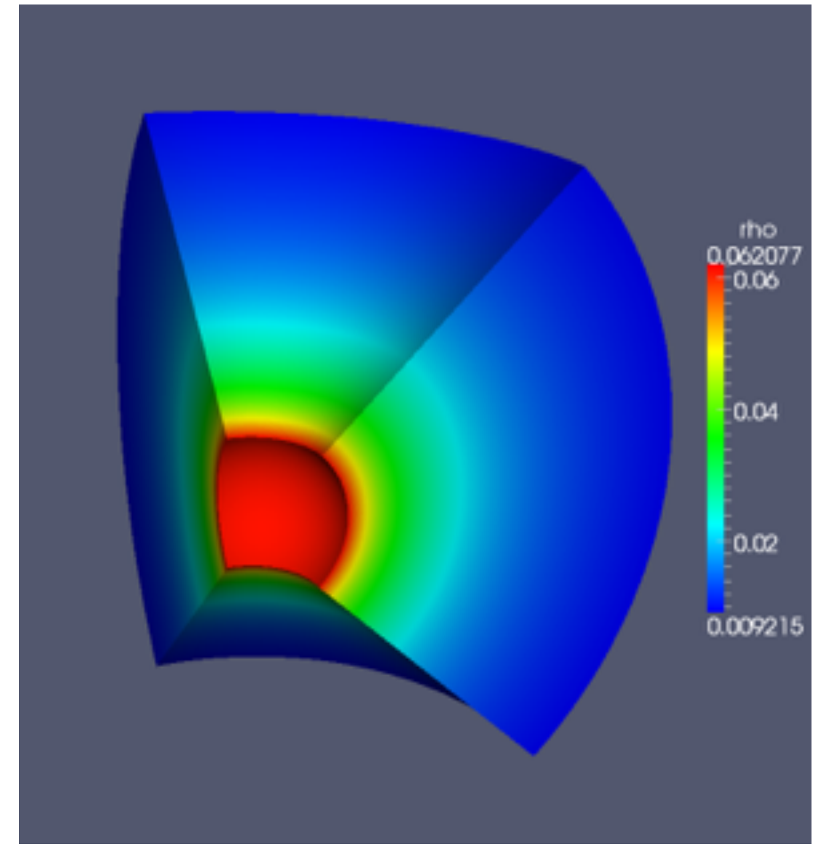
$$\rho^2 = r^2 + a^2 \cos^2\theta,\ \Delta = r^2 - 2Mr + a^2,\ A = (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta$$

Numerical Tests

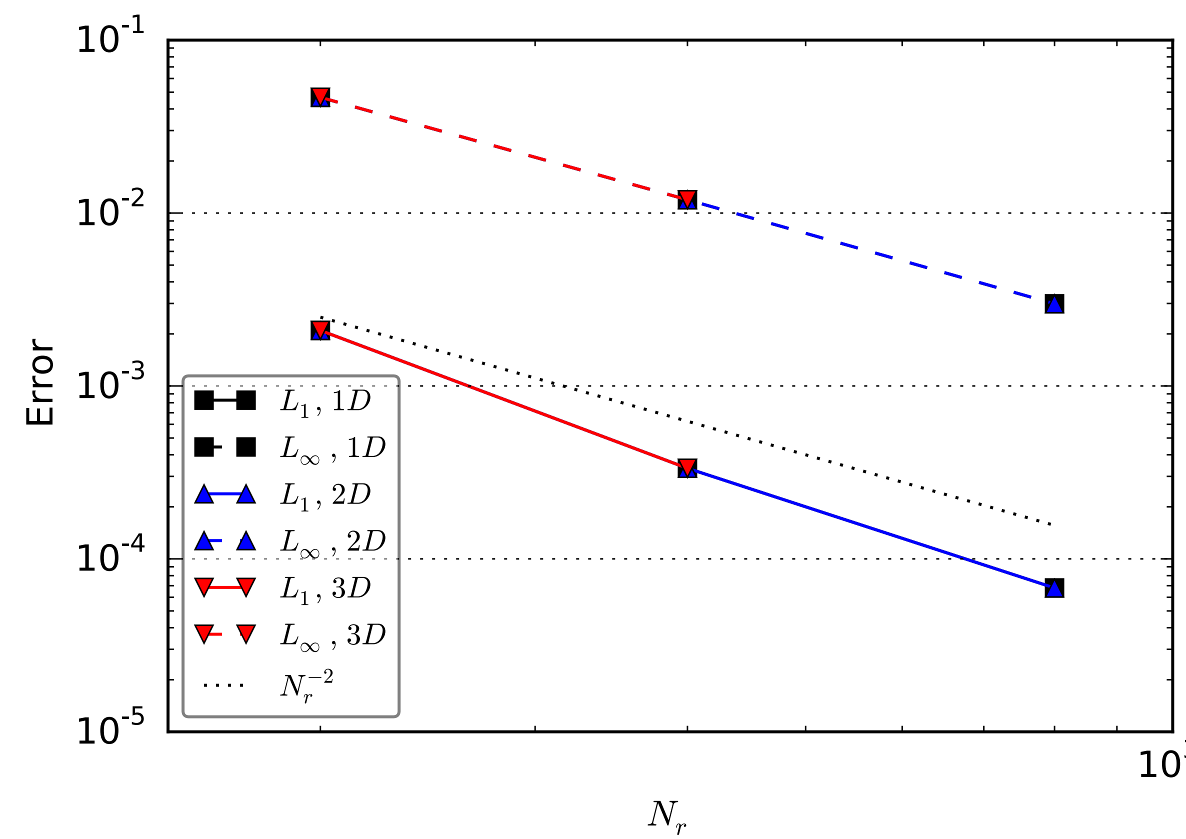
- Various set of numerical tests for validate the code accuracy and performance in SRMHD/GRMHD:
  - Shock-Tube (comp. exact solution, check convergence) in SRMHD
  - Magnetic loop advection (check div. B problem) in SRMHD
  - blast wave propagation w./w.o. Magnetic field in SRMHD
  - Jet propagation in SRMHD
  - Magnetic reconnection (checking resistivity) in SRRMHD
  - Spherical (Bondi) accretion in GRHD/GRMHD
  - Stationary hydrodynamic torus in GRHD
  - Magnetized torus with toroidal B-field in GRMHD

Bondi Accretion



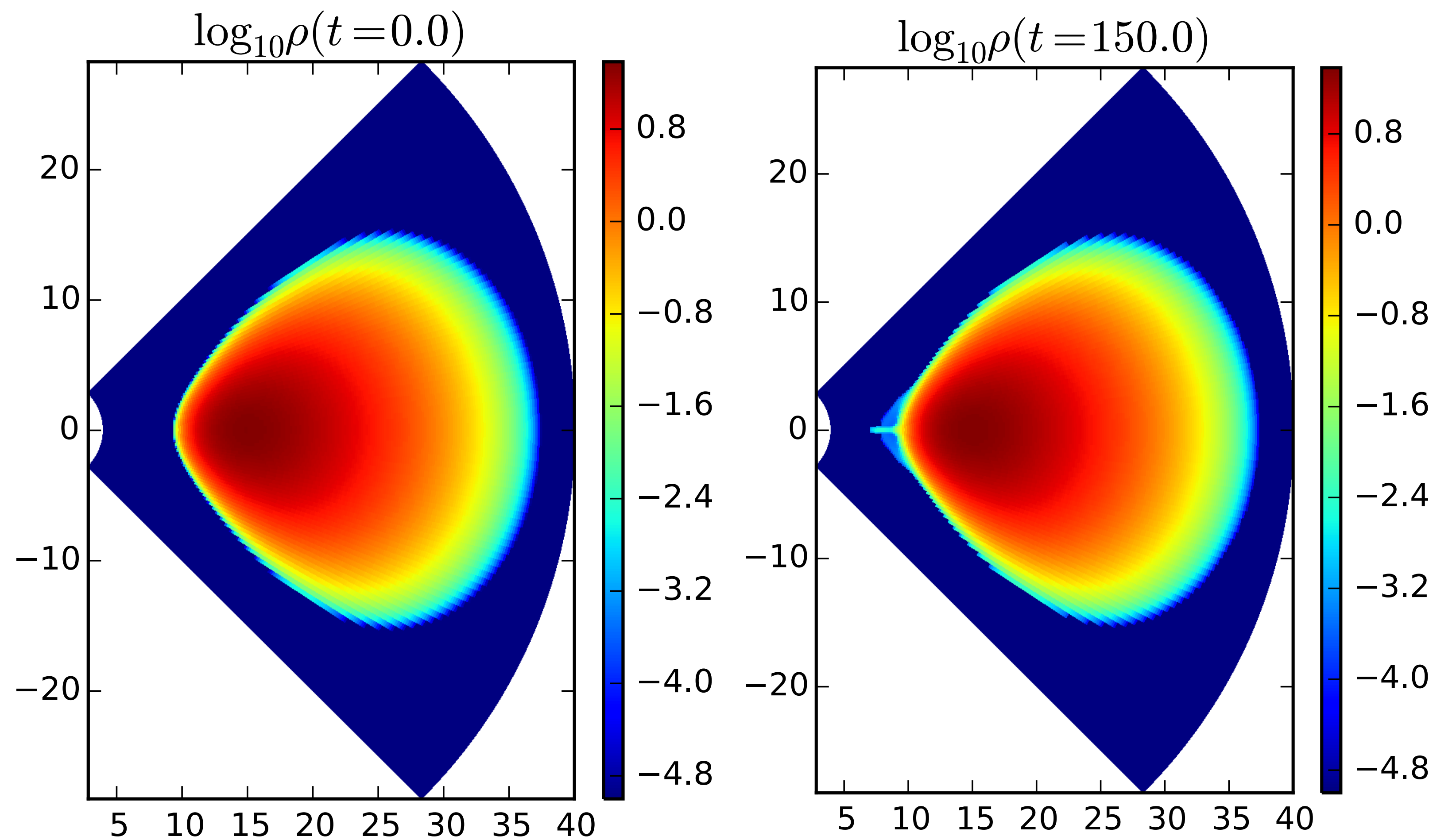


1D Magnetized Bondi accretion in GLM-GRMHD using Schwarzschild coordinates with $r_c = 8$, $\beta_c = 1$. The black solid line shows the initial primitive variables and the symbols the state at t = 100M.



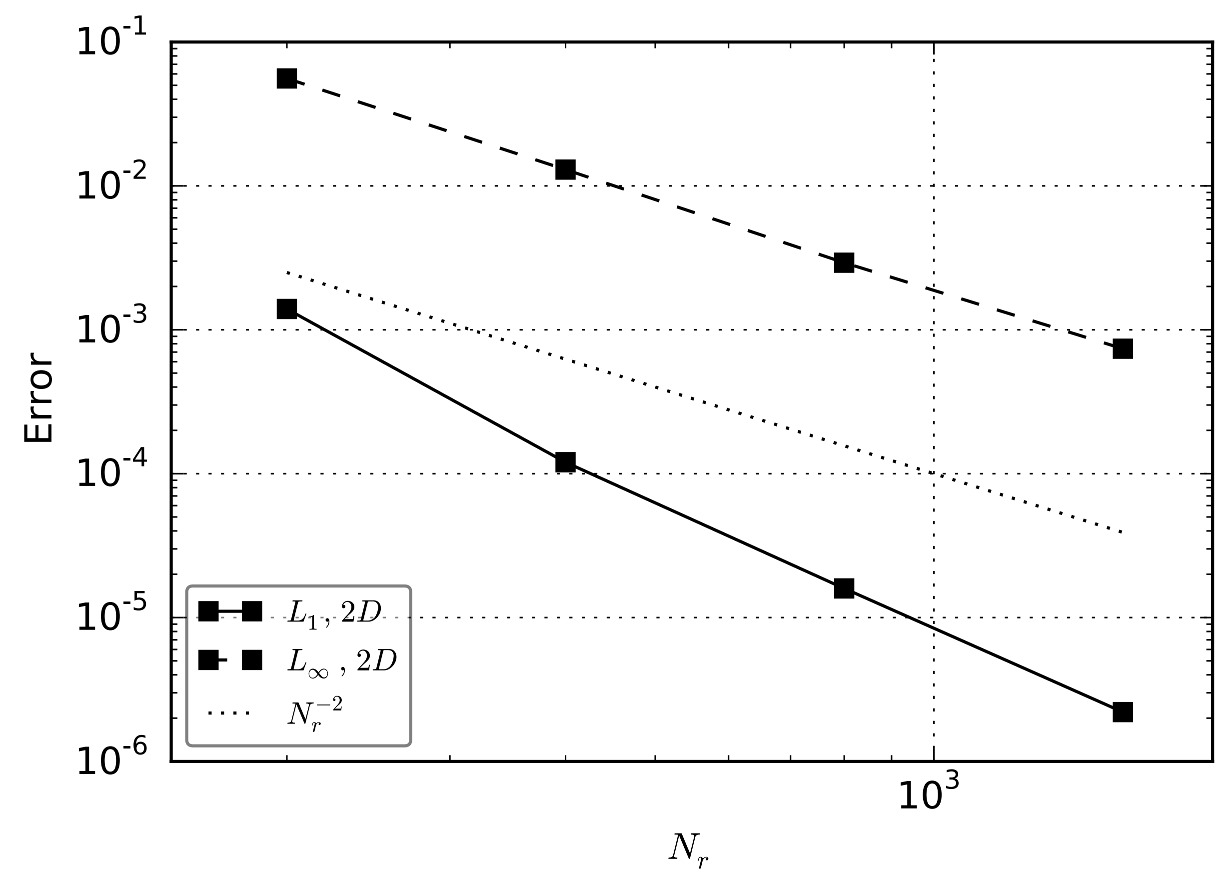
- Bondi accretion is analytical solution of spherical accretion flow onto Schwarzschild BH (Hawley et al. 1984)
- Pure radial B-field does not affect flow structure

Stationary Hydro torus



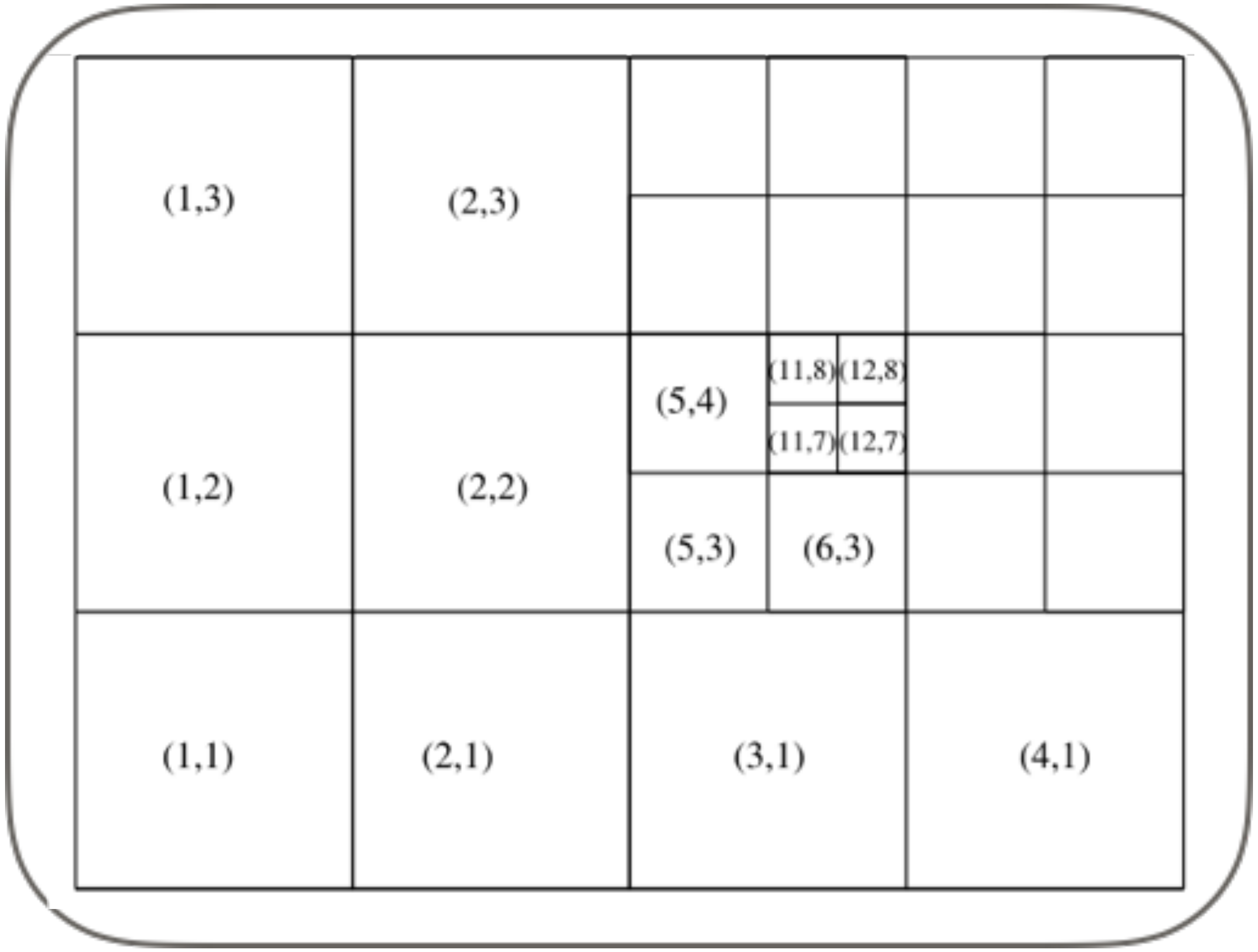
- There are several solutions for stationary hydro-equilibrium torus with constant or different angular momentum (Fishbone & Moncrief 1976, Hawley et al. 1984, Font & Daigne 2002 etc.)

- 2D/3D stationary hydro torus in Kerr-Schild coordinates



Adaptive Mesh

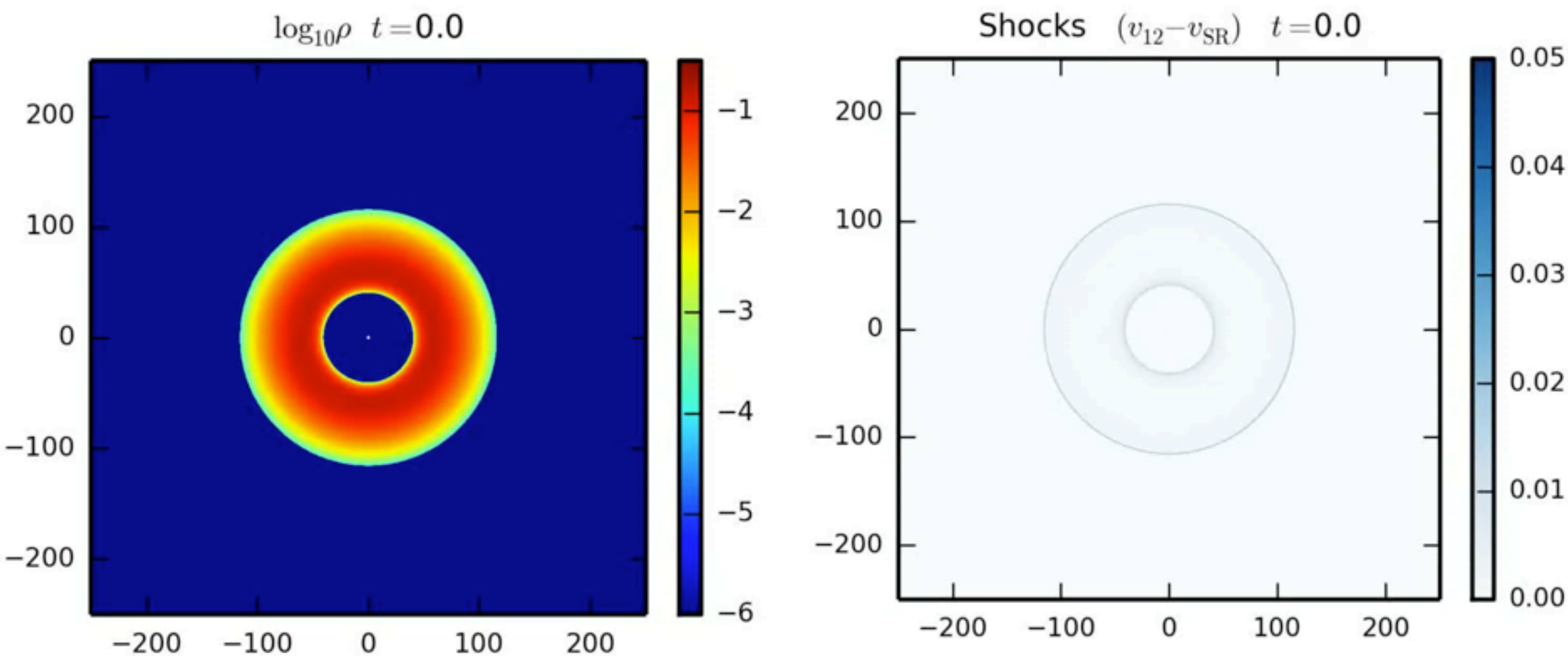
- two-, quad-, or oct- tree AMR: Split the domain in blocks, e.g. 10x10x10 cells.
- The mesh is a “forest”: Each base block has branches pointing to higher levels. A leaf is a computational block.
- A space-filling curve (Morton/Z-order) uniquely runs over the blocks for integration
- Load-balancing is done by cutting the space-filling curve



Fairly standard: Paramesh, Athena++…

AMR Performance in GRHD: recoiling BH

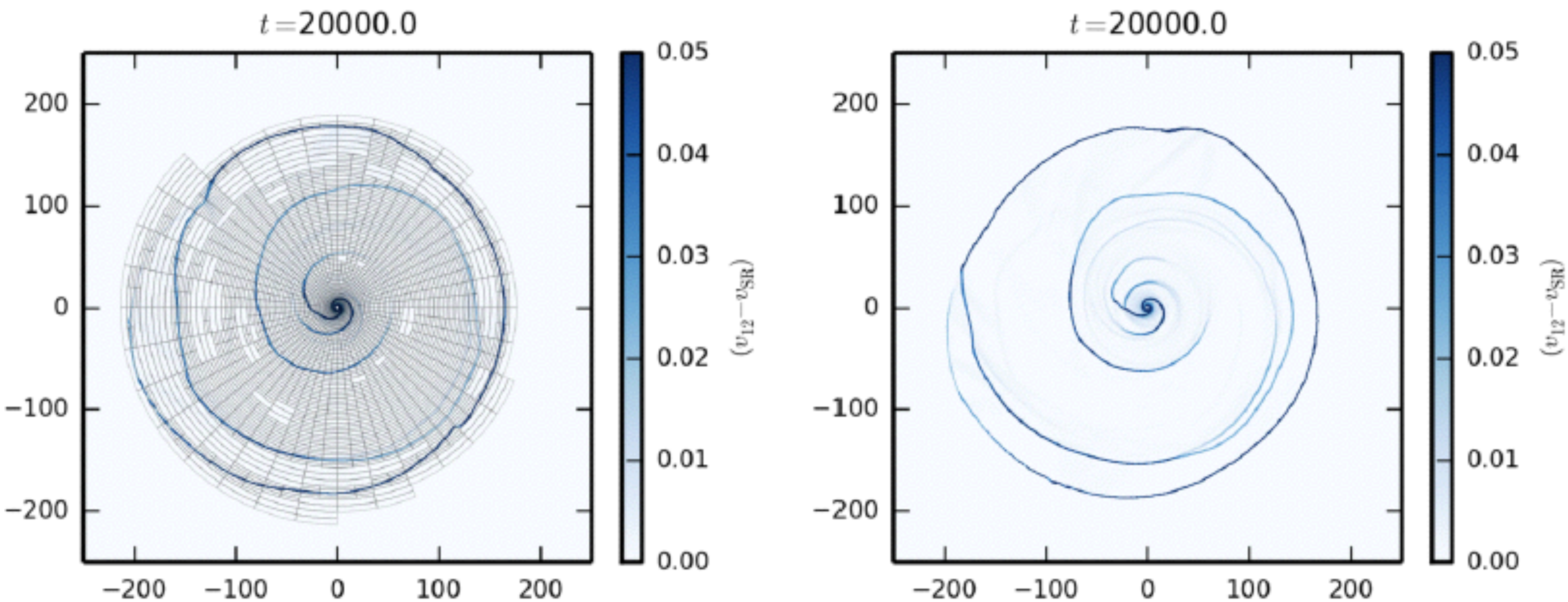
Meliani et al.(2017)



- 2D/3D GRHD simulations of recoiling BH with sub-Keplerian disk produced by the merger of supermassive BH binary
- Asymmetry induced by kicked (recoiled) disk and spiral shock structure is developed.
- AMR is functionally worked for spiral shock region.

AMR Performance in GRHD: recoiling BH

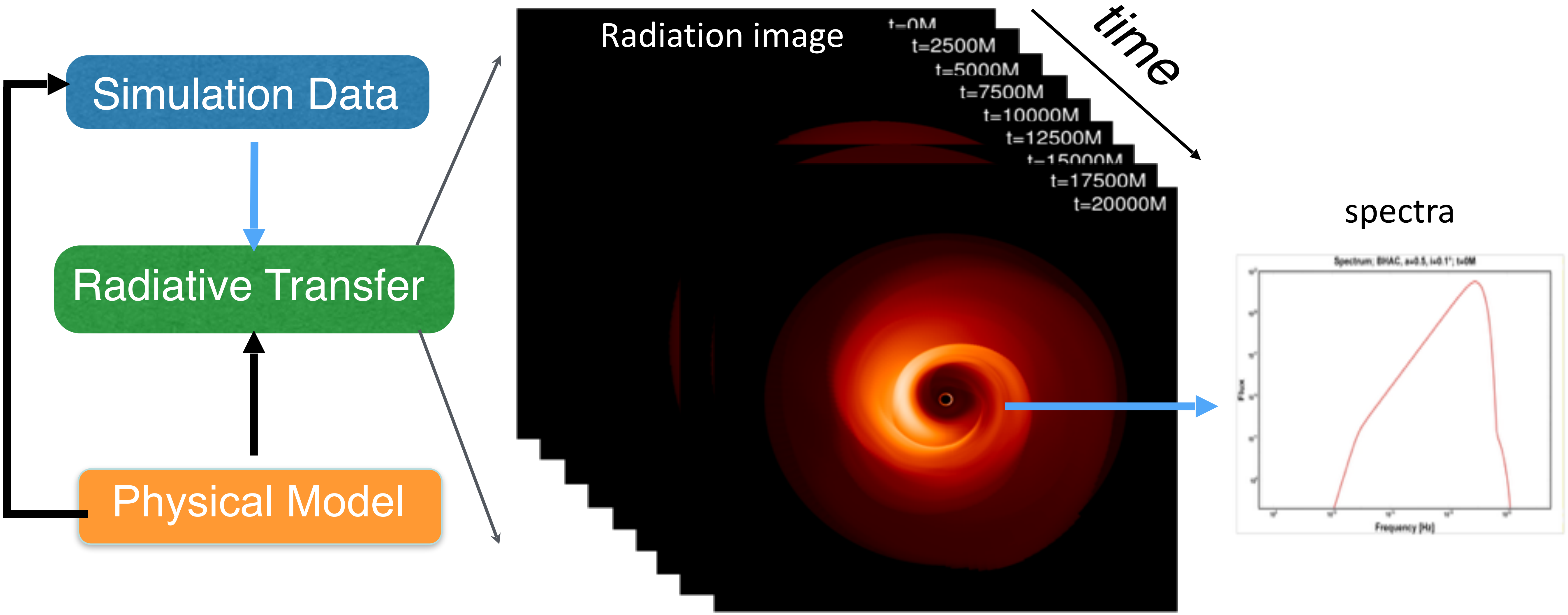
Meliani et al.(2017)



- 2D/3D GRHD simulations of recoiling BH with sub-Keplerian disk produced by the merger of supermassive BH binary
- Asymmetry induced by kicked (recoiled) disk and spiral shock structure is developed.
- AMR is functionally worked for spiral shock region.

Coupling with GRRT code

- GR radiation transfer calculation is using ray-tracing method
- Including thermal & non-thermal radiation process
- Fully-coupled with BHAC & RAISHIN codes (made pipeline)

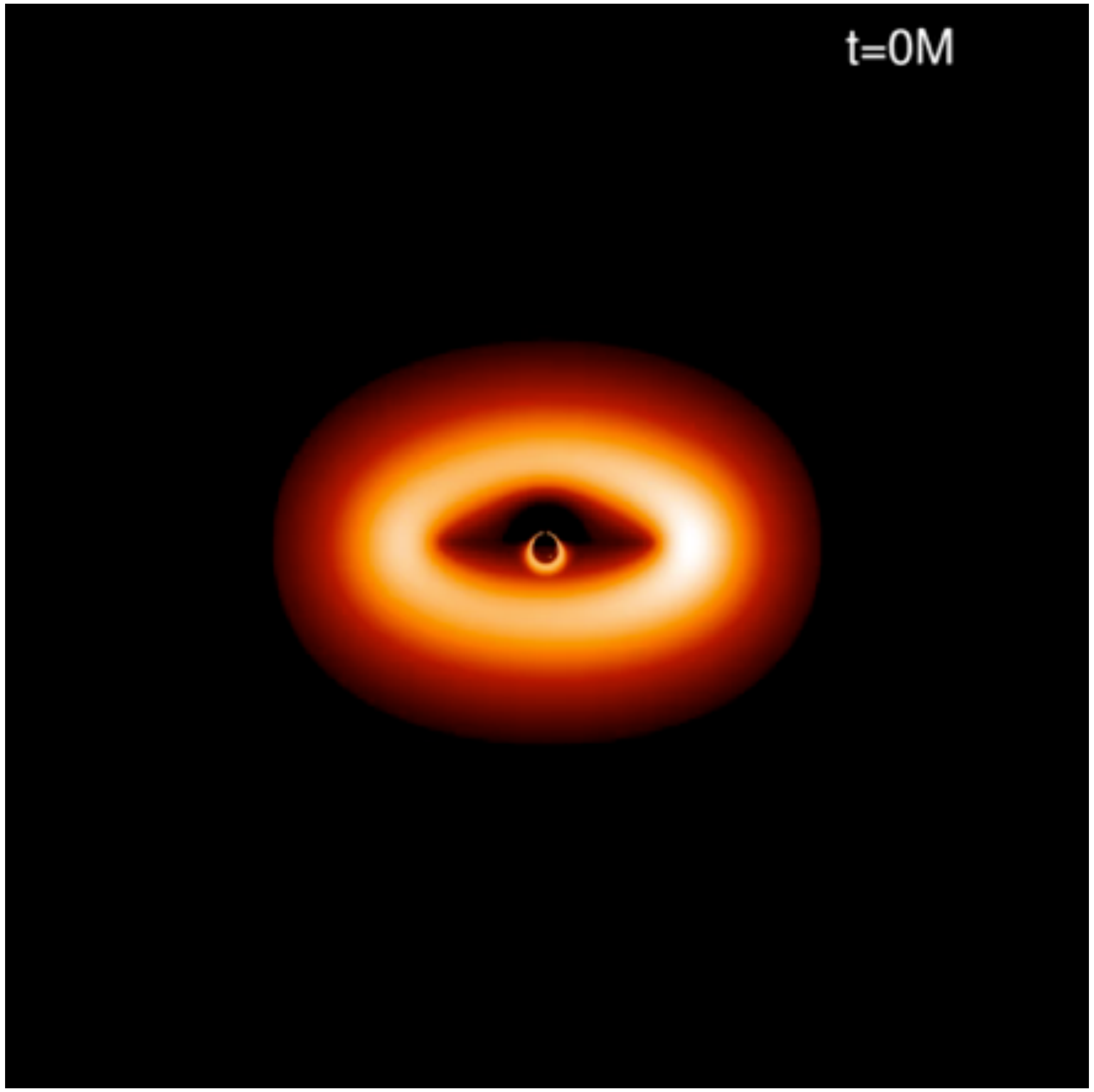


- Combining multiple images, we can make a movie
- Integrating over all images we may also contract the lightcurve

Radiation Image in 3D Recoiling BH

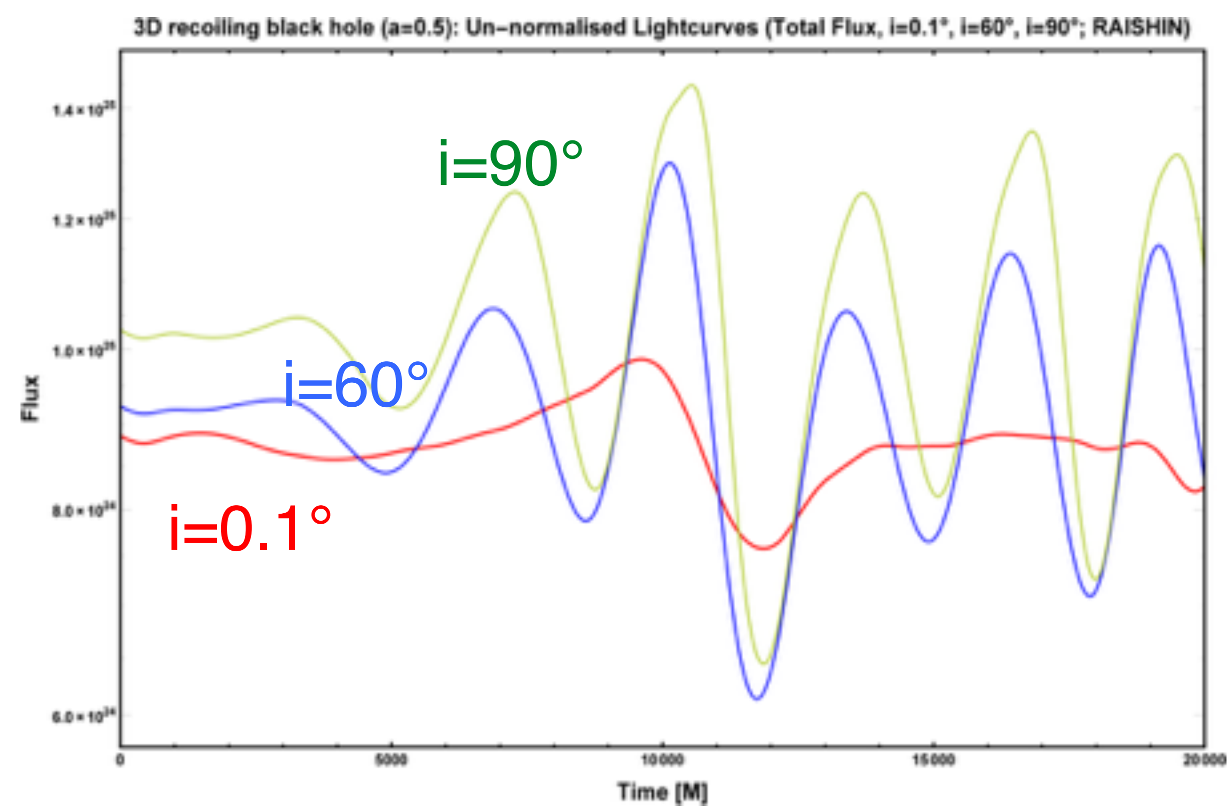
i=60 deg
Total intensity

Meliani et al.(2017)



- Based on 3D GRHD simulation of recoiling BH by BHAC
- Only consider thermal radiation (because no B-field)
- Calculations are performed at multiple observer frequencies –e.g., radio, optical, IR, X-ray

Lightcurve of total flux



Summary

- The GRMHD equations are composed by the conservation laws of density and energy-momentum coupling with Maxwell equations.
- The GRMHD equations describe its dynamics in 4D curved spacetime.
- Solving GRMHD equations numerically, we should use 3+1 formalism to decompose the time and space.
- Valencia formulation is well-used 3+1 form of GRMHD equations.
- Relativistic MHD simulations need additional numerical calculation of conserved variables to primitive variables.