Hydrodynamics and Magnetohydrodynamics: Solutions of the exercises in Lecture X

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Lecture X, Exercise 1.

We start from the conservation equations for energy and linear momentum

$$\nabla_\mu T^{\mu\nu} = 0, \tag{1}$$

where the energy-momentum tensor is given by

$$T^{\mu\nu} = (e+p)u^\mu u^\nu + pg^{\mu\nu}. \tag{2}$$

Assuming for simplicity that the flow is one-dimensional and the spacetime is flat, i.e. $u^\alpha = W(1, v, 0, 0)$, $W = (1 - v^i v_i)^{1/2}$, and $g_{\mu\nu} = \eta_{\mu\nu} = (-1, 1, 1, 1)$, we can rewrite Eq. (1) as

$$\partial_t T^{tt} + \partial_x T^{xt} = 0 \tag{3}$$
$$\partial_t T^{tx} + \partial_x T^{xx} = 0 \tag{4}$$

The relevant components of the energy-momentum tensor are given by

$$\begin{aligned}
T^{tt} &= (e+p)u^t u^t + pg^{tt} \\
&= (e+p)W^2 - p \\
&= W^2(e + pv^2) && (5) \\
T^{tx} &= (e+p)u^t u^x \\
&= (e+p)W^2 v && (6) \\
T^{tt} &= (e+p)u^x u^x + pg^{xx} \\
&= (e+p)W^2 v^2 + p \\
&= ev^2 W^2 + pW^2. && (7)
\end{aligned}$$

As a result, Eqs (3) and (4) are written as

$$\partial_t[(e + pv^2)W^2] + \partial_x[(e+p)W^2 v], \tag{8}$$
$$\partial_t[(e+p)W^2 v] + \partial_x[(ev^2 + p)W^2]. \tag{9}$$

Lecture X, Exercise 2.

Assuming the fluid is initially uniform with energy density, pressure and velocity given by e_0, p_0, and v_0, we can introduce the perturbations

$$e = e_0 + \delta e, \quad p = p_0 + \delta p, \quad v = v_0 + \delta v = \delta v, \tag{10}$$

Next we assume that the initial velocity v_0 is zero and insert the perturbations (**??**) in equations (8) and (9) to obtain the linearized hydrodynamical equations where we drop off 2nd-order terms, e.g. $\delta X \delta Y$, and assume that the initial state is static and uniform, i.e., $\partial_t X_0 = \partial_x X_0 = 0$. In this way we obtain

$$\partial_t\{[(e_0 + \delta e) + (p_0 + \delta p)\delta v^2]W^2\} + \partial_x\{[(e_0 + \delta e) + (p_0 + \delta p)]W^2\delta v\} = 0 \tag{11}$$
$$\partial_t(\delta e W^2) + \partial_x(e_0 W^2 \delta v + p_0 W^2 \delta v) = 0 \tag{12}$$
$$W^2 \partial_t \delta e + W^2(e_0 + p_0)\partial_x \delta v = 0 \tag{13}$$
$$\partial_t \delta e + (e_0 + p_0)\partial_x \delta v = 0 \tag{14}$$

$$\partial_t\{[(e_0 + \delta e) + (p_0 + \delta p)W^2\delta v] + \partial_x\{[(e_0 + \delta e)\delta v^2 + (p_0 + \delta p)]W^2\} = 0 \tag{15}$$
$$\partial_t(e_0 W^2 \delta v + p_0 W^2 \delta v) + \partial_x W^2 \delta p = 0 \tag{16}$$
$$W^2(e_0 + p_0)\partial_t \delta v + W^2 \partial_x \delta p = 0 \tag{17}$$
$$(e_0 + p_0)\partial_t \delta v + \partial_x \delta p = 0 \tag{18}$$

Therefore the final set of linearized equations is

$$\partial_t \delta e + (e_0 + p_0)\partial_x \delta v = 0, \tag{19}$$
$$(e_0 + p_0)\partial_t \delta v + \partial_x \delta p = 0. \tag{20}$$

Taking a time derivative in both equations,

$$\partial_t^2 \delta e = -(e_0 + p_0)\partial_x \partial_t \delta v, \tag{21}$$
$$\partial_x^2 \delta p = -(e_0 + p_0)\partial_x \partial_t \delta v. \tag{22}$$

and combining them we obtain

$$\begin{aligned} &\partial_t^2 \delta e - \partial_x^2 \delta p \\ &= \partial_t^2 \delta e - \partial x^2 \left(\frac{\delta p}{\delta e}\delta e\right) \\ &= \partial_t^2 \delta e - c_s^2 \partial x^2 \delta e \\ &= (\partial_t^2 - c_s^2 \partial x^2)\delta e = \Box \delta e = 0. \end{aligned} \tag{23}$$

The one above is a wave equation with speed c_s, which we define to be

$$c_s^2 = \left(\frac{\partial p}{\partial e}\right)_s. \tag{24}$$

In other words, $\pm c_s$ is the speed at which the perturbations propagate as waves in the fluid and where the $\pm$ sign reflects that the waves can propagate in either direction of our one-dimensional space.

Lecture X, Exercise 3.

The continuity and momentum equations can be written as

$$\begin{aligned} \partial_t(\rho W) + \partial_x(\rho W v) &= 0, & (25)\\ W\partial_t(Wv) + Wv\partial_x(Wv) &= -\frac{1}{\rho h}[\partial_x p + W^2 v\partial_t p + W^2 v^2 \partial_x p]. & (26) \end{aligned}$$

Here we introduce the similarity variable $\xi := x/t$. The differential operators are given by

$$\partial_t = -\left(\frac{\xi}{t}\right)\frac{d}{d\xi}, \quad \partial_x = \left(\frac{1}{t}\right)\frac{d}{d\xi}. \tag{27}$$

Using the similarity variable and the differential operators, the equations (25) and (26) are written as

$$-\left(\frac{\xi}{t}\right)\frac{d}{d\xi}(\rho W) + \left(\frac{1}{t}\right)\frac{d}{d\xi}(\rho W v) = 0 \tag{28}$$

$$-\xi\frac{d}{d\xi}(\rho W) + \frac{d}{d\xi}(\rho W v) = 0 \tag{29}$$

$$-\xi\rho\frac{d}{d\xi}W - \xi W\frac{d}{d\xi}\rho + \rho W\frac{d}{d\xi}v + \rho v\frac{d}{d\xi}W + Wv\frac{d}{d\xi}\rho = 0 \tag{30}$$

$$W(v-\xi)\frac{d}{d\xi}\rho + \rho(v-\xi)\frac{d}{d\xi}W + \rho W\frac{d}{d\xi}v = 0 \tag{31}$$

$$W(v-\xi)\frac{d}{d\xi}\rho + \rho W[W^2 v(v-\xi)+1]\frac{d}{d\xi}v = 0 \tag{32}$$

$$(v-\xi)\frac{d}{d\xi}\rho + \rho[W^2 v(v-\xi)+1]\frac{d}{d\xi}v = 0 \tag{33}$$

where we have used $W^2 = 1/(1-v^2)$ and $dW = W^3 v dv$. Similarly, for the other equation we have

$$\begin{aligned} \rho h W\left(\frac{\xi}{t}\right)\frac{d}{d\xi}(Wv) - \rho h W v\left(\frac{1}{t}\right)\frac{d}{d\xi}(Wv) &= \\ \left(\frac{1}{t}\right)\frac{d}{d\xi}p - W^2 v\left(\frac{\xi}{t}\right)\frac{d}{d\xi}p + W^2 v^2\left(\frac{1}{t}\right)\frac{d}{d\xi}p & \\ \rho h W\left(\frac{1}{t}\right)(\xi - v)\frac{d}{d\xi}(Wv) &= \left(\frac{1}{t}\right)(1 - W^2 v\xi + W^2 v^2)\frac{d}{d\xi}p & (34)\\ \rho h W(\xi - v)\left(W\frac{d}{d\xi}v + v\frac{d}{d\xi}W\right) &= (1 - W^2 v\xi + W^2 v^2)\frac{d}{d\xi}p & (35)\\ \rho h W(\xi - v)(W + W^3 v^2)\frac{d}{d\xi}v &= (W^2 - v^2 W^2 - W^2 v\xi + W^2 v^2)\frac{d}{d\xi}p & (36)\\ \rho h W^4(\xi - v)\frac{d}{d\xi}v &= W^2(1 - v\xi)\frac{d}{d\xi}p & (37)\\ \rho h W^2(\xi - v)\frac{d}{d\xi}v &= (1 - v\xi)\frac{d}{d\xi}p. & (38) \end{aligned}$$

As a result we obtain the following ordinary differential equations describing the self-similar flow in the rarefaction wave

$$(v-\xi)\frac{d}{d\xi}\rho + \rho[W^2 v(v-\xi)+1]\frac{d}{d\xi}v = 0, \tag{39}$$

$$\rho h W^2 (v-\xi)\frac{d}{d\xi}v + (1-v\xi)\frac{d}{d\xi}p = 0. \tag{40}$$