Hydrodynamics and Magnetohydrodynamics: Solutions of the exercises in Lecture XII

Yosuke Mizuno

Winter Semester 2018

Lecture XII, Exercise 1.

Recall the definition: $[\![X]\!] := X_a - X_b$.

(i) $\alpha[\![A]\!] = \alpha(A_a - A_b) = \alpha A_a - \alpha A_b$.
$[\![\alpha A]\!] = \alpha_a A_a - \alpha_b A_b$.
Therefore, if $\alpha_a = \alpha_b$ ($\Leftrightarrow [\![\alpha]\!] = 0$), $\alpha[\![A]\!] = [\![\alpha A]\!]$.

(ii) $[\![A + B]\!] = (A_a + B_a) - (A_b + B_b)$
$= A_a - A_b + B_a - B_b = [\![A]\!] + [\![B]\!]$.

(iii) $[\![AB]\!] = (A_a B_a) - (A_b B_b)$.
$[\![A]\!][\![B]\!] = (A_a - A_b)(B_a - B_b) = A_a B_a - A_a B_b - A_b B_a + A_b B_b$.
Therefore $[\![AB]\!] \neq [\![A]\!][\![B]\!]$.

(iv) $[\![A]\!][\![B]\!] = (A_a - A_b)(B_a - B_b) = (B_a - B_b)(A_a - A_b) = [\![B]\!][\![A]\!]$.

(v) $[\![A]\!]^2 = (A_a - A_b)^2 = A_a^2 - 2A_a A_b + A_b^2$.
$[\![A^2]\!] = (A_a^2 - A_b^2)$.
Therefore $[\![A]\!]^2 \neq [\![A^2]\!]$.

Lecture XII, Exercise 2.

From the junction conditions, the velocities on either side of the shock front in terms of the physical state can be written as

$$v_a^2 = \frac{(p_a - p_b)(e_b + p_a)}{(e_a - e_b)(e_a + p_b)} \quad (1)$$

$$v_b^2 = \frac{(p_a - p_b)(e_a + p_b)}{(e_a - e_b)(e_b + p_a)} \quad (2)$$

From these two equations, we can derive following relations

$$\frac{v_a}{v_b} = \frac{e_b + p_a}{e_a + p_b} \quad (3)$$

$$v_a v_b = \frac{p_a - p_b}{e_a - e_b} \quad (4)$$

Now we consider the case of an ultrarelativistic fluid with $p = e/3$ and $c_s = 1/\sqrt{3}$. Then the eq (3) can be written as

$$\frac{v_a}{v_b} = \frac{e_b + e_a/3}{e_a + e_b/3} = \frac{3e_b + e_a}{3e_a + e_b}. \tag{5}$$

Therefore the velocity ahead the shock can be written as

$$v_a = \left(\frac{3e_b + e_a}{3e_a + e_b}\right) v_b. \tag{6}$$

From the assumption of an ultrarelativistic fluid, eq (4) can be expressed as

$$v_a v_b = \frac{e_a - e_b}{3(e_a - e_b)} = \frac{1}{3}. \tag{7}$$

Therefore the velocity behind the shock can be obtained as

$$v_b = \frac{1}{3v_a}. \tag{8}$$

We put eq (8) into eq (6) and obtain

$$v_a = \left(\frac{3e_b + e_a}{3e_a + e_b}\right) \frac{1}{3v_a} \tag{9}$$

$$v_a^2 = \frac{1}{3}\left(\frac{3e_b + e_a}{3e_a + e_b}\right). \tag{10}$$

And

$$1 - v_a^2 = 1 - \frac{1}{3}\left(\frac{3e_b + e_a}{3e_a + e_b}\right) = \frac{9e_a + 3e_b - 3e_b - e_a}{9e_a + 3e_b} \tag{11}$$

$$= \frac{8e_a}{9e_a + 3e_b} = \frac{8}{3}\left(\frac{e_a}{3e_a + e_b}\right). \tag{12}$$

Therefore the square of the Lorentz factor relative to the velocity ahead the shock is

$$W_a^2 = \frac{1}{1 - v_a^2} = \frac{3}{8}\left(\frac{3e_a + e_b}{e_a}\right). \tag{13}$$

Similarly using eqs (6) and (8), we can get

$$\frac{1}{3v_b} = \left(\frac{3e_b + e_a}{3e_a + e_b}\right) v_b \tag{14}$$

$$v_b^2 = \frac{3e_a + e_b}{3(3e_b + e_a)}. \tag{15}$$

And

$$1 - v_b^2 = \frac{3(3e_b + e_a) + 3e_a + e_b}{3(3e_b + e_a)} = \frac{8e_b}{3(3e_b + e_a)}. \tag{16}$$

As a result, the square of the Lorentz factor relative to the velocity behind the shock is

$$W_b^2 = \frac{1}{1-v_b^2} = \frac{3}{8}\frac{(3e_a+e_b)}{e_b}. \tag{17}$$

The relative velocity of the fluid ahead and behind the shock is given by

$$v_{ab} = \frac{v_a - v_b}{1 - v_a v_b} = \sqrt{\frac{(p_a-p_b)(e_a-e_b)}{(e_a+p_b)(e_b+p_a)}}. \tag{18}$$

From the assumption of an ultrarelativistic fluid, the eq (18) can be written as

$$\begin{aligned} v_{ab} &= \sqrt{\frac{(e_a/3-e_b/3)(e_a-e_b)}{(e_a+e_b/3)(e_b+e_a/3)}} && (19)\\ &= \sqrt{\frac{(e_a-e_b)(3e_a-3e_b)}{(3e_a+e_b)(3e_b+e_a)}}. && (20)\end{aligned}$$

And

$$\begin{aligned} 1-v_{ab}^2 &= \frac{(3e_a+e_b)(3e_b+e_a)-(e_a-e_b)(3e_a-3e_b)}{(3e_a+e_b)(3e_b+e_a)} && (21)\\ &= \frac{9e_ae_b+3e_a^2+3e_b^2+e_ae_b-(3e_a^2-3e_ae_b-3e_ae_b+3e_b^2)}{(3e_a+e_b)(3e_b+e_a)} && (22)\\ &= \frac{16e_ae_b}{(3e_a+e_b)(3e_b+e_a)}. && (23)\end{aligned}$$

Therefore the Lorentz factor square of the relative velocity is

$$W_{ab}^2 = \frac{1}{1-v_{ab}^2} = \frac{(3e_a+e_b)(3e_b+e_a)}{16e_ae_b} = \frac{4}{9}W_a^2W_b^2, \tag{24}$$

where

$$W_a^2W_b^2 = \frac{9}{64}\frac{(3e_a+e_b)(3e_b+e_a)}{e_ae_b}. \tag{25}$$

Lecture XII, Exercise 3.

From eqs (13) and (17),

$$\begin{aligned} W_a^2+W_b^2 &= \frac{3}{8}\frac{(3e_a+e_b)e_b+(3e_b+e_a)e_a}{e_ae_b} && (26)\\ &= \frac{3(3e_ae_b+e_b^2+3e_ae_b+e_a^2)}{8e_ae_b} && (27)\\ &= \frac{3e_a^2+18e_ae_b+3e_b^2}{8e_ae_b}. && (28)\end{aligned}$$

From eq (24), the square of the Lorentz factor of the relative velocity can be written as

$$W_{ab}^2 = \frac{9e_a e_b + 3e_a^2 + 3e_b^2 + e_a e_b}{16 e_a e_b} \tag{29}$$

$$= \frac{3e_a^2 + 10 e_a e_b + 3e_b^2}{16 e_a e_b}. \tag{30}$$

Finally, from eqs (28) and (30) we deduce that

$$W_a^2 - 2W_{ab}^2 + W_b^2 = \frac{3e_a^2 + 18 e_a e_b + 3e_b^2 - (3e_a^2 + 10 e_a e_b + 3e_b^2)}{8 e_a e_b} \tag{31}$$

$$= \frac{8 e_a e_b}{8 e_a e_b} \tag{32}$$

$$= 1. \tag{33}$$