Hydrodynamics and Magnetohydrodynamics: Solutions of the exercises in Lecture XVI

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Winter Semester 2018

Lecture XVI, Exercise 1.

The induction equation in the ideal-MHD limit is given by

$$\partial_t \vec{\boldsymbol{B}} = \vec{\nabla} \times (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}). \tag{1}$$

Using vector identity

$$\vec{\nabla} \times (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}) = \vec{\boldsymbol{A}}(\vec{\nabla} \cdot \vec{\boldsymbol{B}}) - \vec{B}(\vec{\nabla} \cdot \vec{\boldsymbol{A}}) + (\vec{\boldsymbol{B}} \cdot \vec{\nabla})\vec{\boldsymbol{A}} - (\vec{\boldsymbol{A}} \cdot \vec{\nabla})\vec{\boldsymbol{B}},$$

the induction equation (1) can be written as

$$\begin{aligned} \partial_t \vec{\boldsymbol{B}} &= \vec{\boldsymbol{v}}(\vec{\nabla} \cdot \vec{\boldsymbol{B}}) - \vec{\boldsymbol{B}}(\vec{\nabla} \cdot \vec{\boldsymbol{v}}) + (\vec{\boldsymbol{B}} \cdot \vec{\nabla})\vec{\boldsymbol{v}} - (\vec{\boldsymbol{v}} \cdot \vec{\nabla})\vec{\boldsymbol{B}}, && (2) \\ &= -\vec{\boldsymbol{B}}(\vec{\nabla} \cdot \vec{\boldsymbol{v}}) - (\vec{\boldsymbol{v}} \cdot \vec{\nabla})\vec{\boldsymbol{B}} + (\vec{\boldsymbol{B}} \cdot \vec{\nabla})\vec{\boldsymbol{v}}, && (3) \end{aligned}$$

where in RHS of equation, the first term is related to the ”expansion”, the second term is related to the ”advection” of the magnetic field, and the third term corresponds instead to the ”stretching” of the magnetic-field lines along the direction of motion of the plasma.

Lecture XVI, Exercise 2.

The thesis of the frozen-flux theorem is that the flux of magnetic field across an open surface S is conserved, i.e.,

$$\frac{d}{dt}\Phi_{\vec{\boldsymbol{B}}} = 0, \tag{4}$$

where

$$\Phi_{\vec{\boldsymbol{B}}} = \int_S \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{n}}\, ds, \tag{5}$$

is the flux of magnetic field across the open surface S of local norm $\vec{\boldsymbol{n}}$. In order to prove this theorem, we consider a closed loop of fluid element l at two instants in time t and $t + \Delta t$. There are two surfaces S_1 and S_2 which have a loop $l(t)$ and $l(t + \Delta t)$ respectively. The fluid motion between the two time instants of the elements making up l generates a cylinder with the surface S_3. Let's $\Phi_{\vec{\boldsymbol{B}}}$ be the flux enclosed by l, and $\Phi_{\vec{\boldsymbol{B}}_1}$, $\Phi_{\vec{\boldsymbol{B}}_2}$, and $\Phi_{\vec{\boldsymbol{B}}_3}$ be the flux through surface S_1, S_2, and S_3, respectively.

The time derivative of the magnetic flux $\Phi_{\vec{\boldsymbol{B}}}$ is given by

$$\frac{d}{dt}\Phi_{\vec{\boldsymbol{B}}} = \lim_{\Delta t \to 0} \left(\frac{\Phi_{\vec{\boldsymbol{B}}_2}(t + \delta t) - \Phi_{\vec{\boldsymbol{B}}_1}(t)}{\Delta t} \right). \tag{6}$$

From $\vec{\nabla} \cdot \vec{\boldsymbol{B}} = 0$, the net flux through the surfaces at any time is zero. It means that

$$-\Phi_{\vec{\boldsymbol{B}}_1}(t+\ \Delta t) + \Phi_{\vec{\boldsymbol{B}}_2}(t+\ \Delta t) + \Phi_{\vec{\boldsymbol{B}}_3}(t+\ \Delta t) = 0. \tag{7}$$

We can eliminate $\Phi_{\vec{\boldsymbol{B}}_2}(t+\ \Delta t)$ and use definition of flux in expressing $\Phi_{\vec{\boldsymbol{B}}_1}$ and $\Phi_{\vec{\boldsymbol{B}}_3}$

$$\frac{d\Phi_{\vec{\boldsymbol{B}}}}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\int_{S_1} (\vec{\boldsymbol{B}}(t+\ \Delta t) - \vec{\boldsymbol{B}}(t)) \cdot \vec{\boldsymbol{n}}\, dS - \int_{S_3} \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{n}}\, dS \right] \tag{8}$$

The first term on the RHS can be written as

$$\int_{S_1} (\vec{\boldsymbol{B}}(t+\ \Delta t) - \vec{\boldsymbol{B}}(t)) \cdot \vec{\boldsymbol{n}}\, dS = \int_{S_1} \frac{\partial \vec{\boldsymbol{B}}}{\partial t} \cdot \vec{\boldsymbol{n}}\, dS\, \Delta t \tag{9}$$

The area element for S_3 can be written $\vec{\boldsymbol{n}}\, dS = (\vec{\boldsymbol{dl}} \times \vec{\boldsymbol{v}})\, \Delta t$, so that the second term on the RHS can be expressed as

$$\begin{aligned} \int_{S_3} \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{n}}\, dS &= \oint_{l(t)} \vec{\boldsymbol{B}} \cdot (\vec{\boldsymbol{dl}} \times \vec{\boldsymbol{v}})\, \Delta t & (10) \\ &= \oint_{l(t)} (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \cdot \vec{\boldsymbol{dl}}\, \Delta t. & (11) \end{aligned}$$

From Stokes' theorem, the line integral can be changed to a surface integral, i.e.,

$$\int_{S_3} \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{n}}\, dS = \int_{S_1} \vec{\nabla} \times (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \cdot \vec{\boldsymbol{n}}\, dS\, \Delta t. \tag{12}$$

so that, using eqs (9) and (12), equation (8) can be written as

$$\begin{aligned} \frac{d\Phi_{\vec{\boldsymbol{B}}}}{dt} &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\int_{S_1} \frac{\partial \vec{\boldsymbol{B}}}{\partial t} \cdot \vec{\boldsymbol{n}}\, dS\, \Delta t - \int_{S_1} \vec{\nabla} \times (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \cdot \vec{\boldsymbol{n}}\, dS\, \Delta t \right] & (13) \\ &= \int_{S_1} \left[\frac{\partial \vec{\boldsymbol{B}}}{\partial t} - \vec{\nabla} \times (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \right] \cdot \vec{\boldsymbol{n}}\, dS = 0. & (14) \end{aligned}$$

Thus, we reach the conclusion that $\Phi_{\vec{\boldsymbol{B}}}$ does not change in time.

Lecture XVI, Exercise 3.

Here we derive the conserved form of total energy equation in ideal MHD limit. First, the equation of motion can be written as

$$\begin{aligned} \rho \left[\frac{\partial \vec{\boldsymbol{v}}}{\partial t} + (\vec{\boldsymbol{v}} \cdot \vec{\nabla}) \vec{\boldsymbol{v}} \right] + \vec{\nabla} p - \vec{\boldsymbol{j}} \times \vec{\boldsymbol{B}} &= 0 & (15) \\ \rho \vec{\boldsymbol{v}} \left[\frac{\partial \vec{\boldsymbol{v}}}{\partial t} + (\vec{\boldsymbol{v}} \cdot \vec{\nabla}) \vec{\boldsymbol{v}} \right] + \vec{\boldsymbol{v}} \cdot \vec{\nabla} p - \vec{\boldsymbol{v}} \cdot (\vec{\boldsymbol{j}} \times \vec{\boldsymbol{B}}) &= 0 & (16) \\ \frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 \right) - \frac{1}{2} v^2 \frac{\partial \rho}{\partial t} + \frac{1}{2} \rho \vec{\boldsymbol{v}} \cdot \vec{\nabla} v^2 + \vec{\boldsymbol{v}} \cdot \vec{\nabla} p - \vec{\boldsymbol{v}} \cdot (\vec{\boldsymbol{j}} \times \vec{\boldsymbol{B}}) &= 0 & (17) \\ \text{(using continuity equation)} & \\ \frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 \right) + \vec{\nabla} \cdot \left(\frac{1}{2} \rho v^2 \vec{\boldsymbol{v}} \right) + \vec{\boldsymbol{v}} \cdot \vec{\nabla} p - \vec{\boldsymbol{v}} \cdot (\vec{\boldsymbol{j}} \times \vec{\boldsymbol{B}}) &= 0 & (18) \end{aligned}$$

Second, from the energy equation (pressure equation which derived from conservation of entropy), we can rewritten it as

$$\frac{\partial p}{\partial t} + (\vec{\boldsymbol{v}} \cdot \vec{\nabla})p + \gamma p \vec{\nabla} \cdot \vec{\boldsymbol{v}} = 0 \quad (19)$$

(using the ideal-fluid EOS $p = (\gamma - 1)\rho\epsilon$ and the continuity eq.)

$$\frac{\partial \epsilon}{\partial t} + (\vec{\boldsymbol{v}} \cdot \vec{\nabla})\epsilon + (\gamma - 1)\epsilon \vec{\nabla} \cdot \vec{\boldsymbol{v}} = 0 \quad (20)$$

$$\rho \frac{\partial \epsilon}{\partial t} + (\vec{\boldsymbol{v}} \cdot \vec{\nabla})\rho\epsilon + (\gamma - 1)\rho\epsilon \vec{\nabla} \cdot \vec{\boldsymbol{v}} = 0 \quad (21)$$

$$\frac{\partial}{\partial t}(\rho\epsilon) - \epsilon \frac{\partial \rho}{\partial t} + \rho(\vec{\boldsymbol{v}} \cdot \vec{\nabla})\epsilon + p(\vec{\nabla} \cdot \vec{\boldsymbol{v}}) = 0 \quad (22)$$

(using continuity eq.)

$$\frac{\partial}{\partial t}(\rho\epsilon) + \vec{\nabla} \cdot (\rho\epsilon\vec{\boldsymbol{v}}) + p\vec{\nabla} \cdot \vec{\boldsymbol{v}} = 0 \quad (23)$$

Third, the induction equation can be expressed it as

$$\frac{\partial \vec{\boldsymbol{B}}}{\partial t} - \vec{\nabla} \times (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) = 0 (24)$$

$$\vec{\boldsymbol{B}} \cdot \frac{\partial \vec{\boldsymbol{B}}}{\partial t} - \vec{\boldsymbol{B}} \cdot \vec{\nabla} \times (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) = 0 (25)$$

(using the vector identity: $\vec{\nabla} \cdot (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}) = \vec{\boldsymbol{B}} \cdot \vec{\nabla} \times \vec{\boldsymbol{A}} - \vec{\boldsymbol{A}} \cdot \vec{\nabla} \times \vec{\boldsymbol{B}}$)

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2}\right) + \vec{\nabla} \cdot [\vec{\boldsymbol{B}} \times (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}})] - (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \cdot (\vec{\nabla} \times \vec{\boldsymbol{B}}) = 0 (26)$$

(using the vector identity $\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}} = \vec{\boldsymbol{B}}(\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{C}}) - \vec{\boldsymbol{C}}(\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}})$)

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2}\right) + \vec{\nabla} \cdot [(\vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{B}})\vec{\boldsymbol{v}} - (\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{B}})\vec{\boldsymbol{B}}] - (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \cdot \vec{\boldsymbol{j}} = 0 (27)$$

(recalling that $\vec{\nabla} \times \vec{\boldsymbol{B}} = \vec{\boldsymbol{j}}$)

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2}\right) + \vec{\nabla} \cdot [B^2\vec{\boldsymbol{v}} - (\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{B}})\vec{\boldsymbol{B}}] - \vec{\boldsymbol{v}} \cdot (\vec{\boldsymbol{j}} \times \vec{\boldsymbol{B}}) = 0 (28)$$

Using equations (18), (23), and (28), we can obtain the total-energy conservation equation as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\epsilon + \frac{B^2}{2}\right) + \vec{\nabla} \cdot \left[\left(\frac{1}{2}\rho v^2 + \rho\epsilon + \frac{B^2}{2}\right)\vec{\boldsymbol{v}} - (\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{B}})\vec{\boldsymbol{B}}\right]. \quad (29)$$

The terms including magnetic field are the new terms arising in MHD. Clearly, when the magnetic field is zero, equation (29) reduces to the equation of conservation of the total energy in hydrodynamics.

Lecture XVI, Exercise 4.

The set of ideal MHD equations is given by

$$\begin{aligned}
\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot(\rho\vec{\boldsymbol{v}}) &= 0 &(30)\\
\rho\frac{\partial\vec{\boldsymbol{v}}}{\partial t} + \rho(\vec{\boldsymbol{v}}\cdot\vec{\nabla})\vec{\boldsymbol{v}} &= -\vec{\nabla}p + \vec{\boldsymbol{j}}\times\vec{\boldsymbol{B}} &(31)\\
\frac{\partial p}{\partial t} + (\vec{\boldsymbol{v}}\cdot\vec{\nabla})p &= -\gamma p\vec{\nabla}\cdot\vec{\boldsymbol{v}} &(32)\\
p &= \frac{k_{\mathrm{B}}}{m}\rho T &(33)\\
\frac{\partial\vec{\boldsymbol{B}}}{\partial t} &= \vec{\nabla}\times(\vec{\boldsymbol{v}}\times\vec{\boldsymbol{B}}) &(34)\\
\vec{\nabla}\cdot\vec{\boldsymbol{B}} &= 0, &(35)
\end{aligned}$$

which, as usual, represent the continuity equation, the Euler equation, the energy conservation equation, an equation of state (chosen to the ideal-fluid one), the induction equation and divergence-free constraint from the Maxwell equations.

In order to derive linerarized MHD equations, we need to make several simplifying assumptions: in particular, we consider the initial state (which we indicate as $\vec{\boldsymbol{X}}_0$ to be uniform and time independent, i.e., $\partial\vec{\boldsymbol{X}}_0/\partial t = 0 = \partial\vec{\boldsymbol{X}}_0/\partial x$. We also take the initial velocity to be zero, $\vec{\boldsymbol{v}}_0 = 0$. From Euler equation, the static equilibrium implies

$$0 = -\vec{\nabla}p_0 + \vec{\boldsymbol{j}}_0\times\vec{\boldsymbol{B}}_0 \qquad (36)$$

Similarly, from eq (35) we have

$$\vec{\nabla}\cdot\vec{\boldsymbol{B}}_0 = 0 \qquad (37)$$

Here we introduce the perturbed state vector $\vec{\boldsymbol{X}}_1$, such that all quantities can be expanded as

$$\begin{aligned}
&\rho = \rho_0 + \rho_1, \qquad p = p_0 + p_1, \qquad T = T_0 + T_1, &(38)\\
&\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}_0 + \vec{\boldsymbol{v}}_1, \qquad \vec{\boldsymbol{B}} = \vec{\boldsymbol{B}}_0 + \vec{\boldsymbol{B}}_1, \qquad \vec{\boldsymbol{j}} = \vec{\boldsymbol{j}}_{\boldsymbol{0}} + \vec{\boldsymbol{j}}_1. &(39)
\end{aligned}$$

Introducing these perturbations in the set of equations (30)–(35) and neglecting second order terms we obtain, for instance, that the perturbed continuity equation becomes

$$\begin{aligned}
\frac{\partial\rho_0}{\partial t} + \frac{\partial\rho_1}{\partial t} + \vec{\nabla}\cdot(\rho_0\vec{\boldsymbol{v}}_1) + \vec{\nabla}\cdot(\rho_1\vec{\boldsymbol{v}}_1) &= 0 &(40)\\
\frac{\partial\rho_1}{\partial t} + \vec{\nabla}\cdot(\rho_0\vec{\boldsymbol{v}}_1) &= 0. &(41)
\end{aligned}$$

Similarly, the linearized Euler equation is expressed as

$$\begin{aligned}\rho_0\frac{\partial\vec{\boldsymbol{v}}_1}{\partial t}+\rho_1\frac{\partial\vec{\boldsymbol{v}}_1}{\partial t}+\rho_0(\vec{\boldsymbol{v}}_1\cdot\vec{\nabla})\vec{\boldsymbol{v}}_1+\rho_1(\vec{\boldsymbol{v}}_1\cdot\vec{\nabla})\vec{\boldsymbol{v}}_1 \quad &= \\ -\vec{\nabla}p_0-\vec{\nabla}p_1+\vec{\boldsymbol{j}}_0\times\vec{\boldsymbol{B}}_0+\vec{\boldsymbol{j}}_0\times\vec{\boldsymbol{B}}_1+\vec{\boldsymbol{j}}_1\times\vec{\boldsymbol{B}}_0+\vec{\boldsymbol{j}}_1\times\vec{\boldsymbol{B}}_1 & \end{aligned} \tag{42}$$

$$\rho_0\frac{\partial\vec{\boldsymbol{v}}_1}{\partial t}=-\vec{\nabla}p_1+(\vec{\nabla}\times\vec{\boldsymbol{B}}_0)\times\vec{\boldsymbol{B}}_1+(\vec{\nabla}\times\vec{\boldsymbol{B}}_1)\times\vec{\boldsymbol{B}}_0+(-\vec{\nabla}p_0+\vec{\boldsymbol{j}}_0\times\vec{\boldsymbol{B}}_0) \tag{43}$$

[using eq. (36)]

$$\rho_0\frac{\partial\vec{\boldsymbol{v}}_1}{\partial t}=-\vec{\nabla}p_1+(\vec{\nabla}\times\vec{\boldsymbol{B}}_1)\times\vec{\boldsymbol{B}}_0 \tag{44}$$

Similarly, the linearized energy-conservation equation is

$$\frac{\partial p_0}{\partial t}+\frac{\partial p_1}{\partial t}+(\vec{\boldsymbol{v}}_1\cdot\vec{\nabla})p_0+(\vec{\boldsymbol{v}}_1\cdot\vec{\nabla})p_1=-\gamma p_0\vec{\nabla}\cdot\vec{\boldsymbol{v}}_1-\gamma p_1\vec{\nabla}\cdot\vec{\boldsymbol{v}}_1 \tag{45}$$

$$\frac{\partial p_1}{\partial t}+(\vec{\boldsymbol{v}}_1\cdot\vec{\nabla})p_0=-\gamma p_0\vec{\nabla}\cdot\vec{\boldsymbol{v}}_1. \tag{46}$$

Similarly, the linearized equation of state for an ideal fluid is

$$p_0+p_1=\frac{k_{\mathrm{B}}}{m}\rho_0T_0+\frac{k_{\mathrm{B}}}{m}\rho_1T_0+\frac{k_{\mathrm{B}}}{m}\rho_0T_1+\frac{k_{\mathrm{B}}}{m}\rho_1T_1 \tag{47}$$

(using eq (33))

$$p_1=\frac{k_{\mathrm{B}}}{m}\rho_1T_0+\frac{k_{\mathrm{B}}}{m}\rho_0T_1. \tag{48}$$

Similarly, the linearized induction equation is

$$\frac{\partial\vec{\boldsymbol{B}}_0}{\partial t}+\frac{\partial\vec{\boldsymbol{B}}_1}{\partial t}=\vec{\nabla}\times(\vec{\boldsymbol{v}}_1\times\vec{\boldsymbol{B}}_0)+\vec{\nabla}\times(\vec{\boldsymbol{v}}_1\times\vec{\boldsymbol{B}}_1) \tag{49}$$

$$\frac{\partial\vec{\boldsymbol{B}}_1}{\partial t}=\vec{\nabla}\times(\vec{\boldsymbol{v}}_1\times\vec{\boldsymbol{B}}_0). \tag{50}$$

Finally, the solenoidal constraint can be expressed as

$$\vec{\nabla}\cdot\vec{\boldsymbol{B}}_0+\vec{\nabla}\cdot\vec{\boldsymbol{B}}_1=0=\vec{\nabla}\cdot\vec{\boldsymbol{B}}_1=0. \tag{51}$$

Collecting all equations, we write the set of linearized ideal-MHD equations as

$$\frac{\partial\rho_1}{\partial t}+\vec{\nabla}\cdot(\rho_0\vec{\boldsymbol{v}}_1) = 0, \tag{52}$$

$$\rho_0\frac{\partial\vec{\boldsymbol{v}}_1}{\partial t} = -\vec{\nabla}p_1+(\vec{\nabla}\times\vec{\boldsymbol{B}}_1)\times\vec{\boldsymbol{B}}_0, \tag{53}$$

$$\frac{\partial p_1}{\partial t}+(\vec{\boldsymbol{v}}_1\cdot\vec{\nabla})p_0 = -\gamma p_0\vec{\nabla}\cdot\vec{\boldsymbol{v}}_1, \tag{54}$$

$$p_1 = \frac{k_{\mathrm{B}}}{m}\rho_1T_0+\frac{k_{\mathrm{B}}}{m}\rho_0T_1, \tag{55}$$

$$\frac{\partial\vec{\boldsymbol{B}}_1}{\partial t} = \vec{\nabla}\times(\vec{\boldsymbol{v}}_1\times\vec{\boldsymbol{B}}_0), \tag{56}$$

$$\vec{\nabla}\cdot\vec{\boldsymbol{B}}_1 = 0. \tag{57}$$