Hydrodynamics and Magnetohydrodynamics: Solutions of the exercises in Lecture II

Yosuke Mizuno

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Lecture II, Exercise 1.

The mass conservation equation can be written as

$$\partial_t \rho = -\partial_i(\rho v_i) = -\rho \partial_i v_i - v_i \partial_i \rho. \tag{1}$$

Next, we consider the momentum conservation equation. We expand the derivative and using Eq. 1,

$$\begin{aligned}
& \partial_t(\rho v_j) + \partial_i(\rho v_i v_j) + \partial_i P_{ij} - \frac{\rho}{m} F_j \\
= \;& \rho \partial_t v_j + v_j \partial_t \rho + \rho v_i \partial_i v_j + \rho v_j \partial_i v_i + v_i v_j \partial_i \rho + \partial_i P_{ij} - \frac{\rho}{m} F_j \\
= \;& \rho \partial_t v_j - \rho v_j \partial_i v_i - v_i v_j \partial_i \rho + \rho v_i \partial_i v_j + \rho v_j \partial_i v_i + v_i v_j \partial_i \rho + \partial_i P_{ij} - \frac{\rho}{m} F_j \\
= \;& \rho \partial_t v_j + \rho v_i \partial_i v_j + \partial_i P_{ij} - \frac{\rho}{m} F_j = 0.
\end{aligned} \tag{2}$$

Then it is divided by ρ, we obtain

$$\partial_t v_j + v_i \partial_i v_j + \frac{1}{\rho} \partial_i P_{ij} - \frac{1}{m} F_j = 0. \tag{3}$$

Next, we consider the energy conservation equation. We expand the derivative and using Eq. 1,

$$\begin{aligned}
& \partial_t(\rho \epsilon) + \partial_i(\rho \epsilon v_i) + \partial_i q_i + P_{ij} \Lambda^{ij} \\
= \;& \rho \partial_t \epsilon + \epsilon \partial_t \rho + \rho \epsilon \partial_i v_i + \rho v_i \partial_i \epsilon + \epsilon v_i \partial_i \rho + \partial_i q_i + P_{ij} \Lambda^{ij} \\
= \;& \rho \partial_t \epsilon - \rho \epsilon \partial_i v_i - \epsilon v_i \partial_i \rho + \rho \epsilon \partial_i v_i + \rho v_i \partial_i \epsilon + \epsilon v_i \partial_i \rho + \partial_i q_i + P_{ij} \Lambda^{ij} \\
= \;& \rho \partial_t \epsilon + \rho v_i \partial_i \epsilon + \partial_i q_i + P_{ij} \Lambda^{ij} = 0.
\end{aligned} \tag{4}$$

Finally, dividing by ρ we obtain

$$\partial_t \epsilon + v_i \partial_i \epsilon + \frac{1}{\rho} \partial_i q_i + \frac{1}{\rho} P_{ij} \Lambda^{ij} = 0. \tag{5}$$

Lecture II, Exercise 2.

Prove the following identity:

$$\rho\langle u_i \partial_i |\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2\rangle = 2P_{ij}\Lambda^{ij}. \tag{6}$$

The LHS of Eq. 6 is For the LHS of Eq. 6, we use $\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}} = \vec{\boldsymbol{A}}$. Then it becomes

$$\begin{aligned}
\rho\langle u_i \partial_i (A^j A^k \delta_{jk})\rangle &= \rho\langle u_i [(\partial_i A^j) A^k \delta_{jk} + A^j (\partial_i A^k)\delta_{jk}]\rangle \\
&= 2\rho\langle u_i \partial_i A^j A_j\rangle \\
&= 2\rho\langle u_i [\partial_i (u_j - v_j)](u_j - v_j)\rangle = 2\rho\langle u_i [\partial_i u_j - \partial_i v_j](u_j - v_j)\rangle \\
&= -2\rho\langle u_i \partial_i v_j (u_j - v_j)\rangle = -2\rho\partial_i v_j \langle u_i (u_j - v_j)\rangle
\end{aligned} \tag{7}$$

Next, we reconsider the pressure tensor,

$$\begin{aligned}
P_{ij} &= \rho\langle (u_i - v_i)(u_j - v_j)\rangle \\
&= \rho\langle u_i (u_j - v_j) - v_i (u_j - v_j)\rangle = \rho\langle u_i (u_j - v_j)\rangle - \rho\langle v_i (u_j - v_j)\rangle \\
&= \rho\langle u_i (u_j - v_j)\rangle - \rho[\langle v_i u_j\rangle - \langle v_i v_j\rangle] \\
&= \rho\langle u_i (u_j - v_j)\rangle - \rho[v_i \langle u_j\rangle - v_i v_j] = \rho\langle u_i (u_j - v_j)\rangle.
\end{aligned} \tag{8}$$

Using Eq. 8, Eq. 7 is written as

$$\rho\langle u_i \partial_i |\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2\rangle = -2\rho\partial_i v_j \langle u_i (u_j - v_j)\rangle = -2P_{ij}\partial_i v_j. \tag{9}$$

Here we introduce $\partial_i v_j = A_{ij}$. This is a generic tensor. However, P_{ij} is a symmetric tensor. Hence A_{ij} must also be symmetric tensor.

$$\begin{aligned}
A_{ij} &= \frac{1}{2}(A_{ij} + A_{ji}) \\
&= \frac{1}{2}(\partial_i v_j + \partial_j v_i) = \Lambda_{ij}.
\end{aligned} \tag{10}$$

Using Eq. 11, Eq. 9 can be changed as

$$-2P_{ij}\partial_i v_j = -2P_{ij}\Lambda^{ij}. \tag{11}$$

Therefore

$$\rho\langle u_i \partial_i |\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2\rangle = 2P_{ij}\Lambda^{ij}. \tag{12}$$