Hydrodynamics and Magnetohydrodynamics: Solutions of the exercises in Lecture III

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Lecture III, Exercise 1.

The Maxwell-Boltzmann (equilibrium) distribution function is considered in the following general form

$$\ln(f_0(\vec{\boldsymbol{u}}_0)) = -A(\vec{\boldsymbol{u}} - \vec{\boldsymbol{u}}_0)^2 + \ln C, \tag{1}$$

$$f_0 = C\exp(-A(\vec{\boldsymbol{u}} - \vec{\boldsymbol{u}}_0)^2). \tag{2}$$

We recall the following integral identities.

$$\int_{-\infty}^{\infty} e^{-A\vec{\boldsymbol{u}}^2} d^3u = \frac{\pi^{3/2}}{A^{3/2}}, \quad \int_{-\infty}^{\infty} \vec{\boldsymbol{u}} e^{-A\vec{\boldsymbol{u}}^2} d^3u = 0, \quad \int_{-\infty}^{\infty} \vec{\boldsymbol{u}}^2 e^{-A\vec{\boldsymbol{u}}^2} d^3u = \frac{3}{2}\frac{\pi^{3/2}}{A^{5/2}}. \tag{3}$$

First we consider the number density n. Using Eq.(2),

$$n = \int f d^3u = \int_{-\infty}^{\infty} C e^{-A(\vec{\boldsymbol{u}} - \vec{\boldsymbol{u}}_0)^2} d^3u. \tag{4}$$

Using $\vec{\bar{\boldsymbol{u}}} = (\vec{\boldsymbol{u}} - \vec{\boldsymbol{u}}_0)$ and $d\vec{\bar{\boldsymbol{u}}} = du$, Eq. (3) becomes

$$\int_{-\infty}^{\infty} C e^{-A\vec{\bar{\boldsymbol{u}}}^2} d^3\vec{\bar{\boldsymbol{u}}} = C\left(\frac{\pi^{3/2}}{A^{3/2}}\right). \tag{5}$$

Thus, the constant C can be written as

$$C = n\left(\frac{A}{\pi}\right)^{3/2}. \tag{6}$$

Next, we consider the specific internal energy ϵ. Using Eq. (2),

$$\epsilon = \frac{1}{2}(|\vec{\boldsymbol{u}} - \vec{\boldsymbol{u}}_0|^2) = \frac{1}{2n}\int |\vec{\boldsymbol{u}}\, \vec{\boldsymbol{u}}_0|^2 f d^3u \tag{7}$$

$$= \frac{1}{2n}\int_{-\infty}^{\infty} |\vec{\boldsymbol{u}} - \vec{\boldsymbol{u}}_0|^2 C e^{-A(\vec{\boldsymbol{u}} - \vec{\boldsymbol{u}}_0)^2} d^3u. \tag{8}$$

Using $\vec{\bar{\boldsymbol{u}}} = (\vec{\boldsymbol{u}} - \vec{\boldsymbol{u}}_0)$ and $d\vec{\bar{\boldsymbol{u}}} = du$, Eq. (8) becomes

$$\frac{C}{2n}\int_{-\infty}^{\infty} \vec{\bar{\boldsymbol{u}}}^2 e^{-A\vec{\bar{\boldsymbol{u}}}^2} d^3\vec{\bar{\boldsymbol{u}}} = \frac{3C}{4n\pi}\left(\frac{\pi}{A}\right)^{5/2}. \tag{9}$$

Using Eq. (6), the internal energy can be written as

$$\epsilon = \frac{3}{4n\pi} n \left(\frac{A}{\pi}\right)^{3/2} \left(\frac{\pi}{A}\right)^{5/2} = \frac{3}{4A}. \tag{10}$$

Therefore the constant A can be found to be

$$A = \frac{3}{4\epsilon}. \tag{11}$$

Using Eq. (11), the constant C is also written as

$$C = n \left(\frac{3}{4\pi\epsilon}\right)^{3/2}. \tag{12}$$

Lecture III, Exercise 2.

The specific internal energy is given by

$$\epsilon = \frac{3}{2}\frac{k_B T}{m}. \tag{13}$$

Using Eq. (13), the constant A and C can be obtained

$$A = \frac{3}{4} \cdot \frac{2m}{3k_B T} = \frac{m}{2k_B T}, \tag{14}$$

$$C = n \left(\frac{3}{4\pi} \cdot \frac{2m}{3k_B T}\right)^{3/2} = n \left(\frac{m}{2\pi k_B T}\right)^{3/2}. \tag{15}$$

Next, we put these two quantities into the general form of the Maxwell-Boltzmann (equilibrium) distribution function Eq. (2). In this way we obtain

$$f_0(\vec{\boldsymbol{u}}) = n \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{m(\vec{\boldsymbol{u}} - \vec{\boldsymbol{u}}_0)^2}{2k_B T}\right). \tag{16}$$

Lecture III, Exercise 3.

Using a set of spherical coordinates (r, θ, ϕ) in the velocity space, any distribution function can be written as ($du_x du_y du_z = u^2 \sin\theta du d\theta d\phi$)

$$\int_{-\infty}^{\infty} f(u) d^3u = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{\infty} u^2 f(u) du = \int_0^{\infty} 4\pi u^2 f(u) du. \tag{17}$$

Next, we use the Maxwell-Boltzmann distribution function with zero macroscopic velocity (i.e., $\vec{\boldsymbol{v}} = 0$), which is given by

$$f = 4\pi n u^2 \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{mu^2}{2k_B T}\right). \tag{18}$$

The average speed v can be calculated as

$$\begin{aligned} v = \langle u \rangle &= \frac{1}{n}\int_{-\infty}^{\infty} \vec{\boldsymbol{u}} f d^3u \\ &= \frac{1}{n}\int_{0}^{\infty} 4\pi u^2 n \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(\frac{-mu^2}{2k_BT}\right) u du. \\ &= 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_0^\infty u^3 \exp\left(\frac{-mu^2}{2k_BT}\right) du. \end{aligned} \quad (19)$$

Now we introduce a new parameter $x = u^2$, so that taking a derivative we obtain

$$\frac{d}{du}u^2 = \frac{dx}{du}, \quad (20)$$

$$du = \frac{1}{2u}dx. \quad (21)$$

The integral part of Eq. (19) becomes

$$\begin{aligned} &\int_0^\infty u^3 \exp\left(-\frac{mu^2}{2k_BT}\right) du = \frac{1}{2}\int_0^\infty x \exp\left(-\frac{mx}{2k_BT}\right) dx \\ &= \frac{1}{2}\int_0^\infty x \left[-\frac{2k_BT}{m}\exp\left(-\frac{2k_BT}{m}\right)'\right] dx \\ &= \frac{1}{2}\left[x\left\{-\frac{2k_BT}{m}\exp\left(-\frac{mx}{2k_BT}\right)\right\}\right]_0^\infty - \frac{1}{2}\int_0^\infty (x)'\left[-\frac{2k_BT}{m}\exp\left(-\frac{mx}{2k_BT}\right)\right] dx \\ &= -\frac{k_BT}{m}\left[x\exp\left(-\frac{mx}{2k_BT}\right)\right]_0^\infty + \frac{k_BT}{m}\int_0^\infty \exp\left(-\frac{mx}{2k_BT}\right) dx \\ &= 0 + \frac{k_BT}{m}\left[-\frac{2k_BT}{m}\exp\left(-\frac{mx}{2k_BT}\right)\right]_0^\infty \\ &= -\frac{2k_B^2T^2}{m^2}(0-1) = \frac{2k_B^2T^2}{m^2}. \end{aligned} \quad (22)$$

Therefore the average velocity is obtained to be

$$\begin{aligned} v &= 4\pi\left(\frac{m}{2\pi k_BT}\right)^{3/2} \cdot \frac{2k_B^2T^2}{m^2} \\ &= \frac{4k_BT}{m}\sqrt{\frac{m}{2\pi k_BT}} = \sqrt{\frac{8k_BT}{\pi m}}. \end{aligned} \quad (23)$$

Lecture III, Exercise 4.

The most probable speed is the maximum of Maxwell-Boltzmann distribution function, that is, the one for which $df/du = 0$. We use the Maxwell-Boltzmann distribution function with zero macroscopic velocity in spherical coordinates (i.e., Eq. (18)).

Taking a derivative of Eq (18)

$$\frac{df}{du} = \left[4\pi n \left(\frac{m}{2\pi k_B T}\right)^{3/2}\right] \left[2u \exp\left(-\frac{mu^2}{2k_B T}\right) + u^2 \left(-\frac{2mu}{2k_B T}\right) \exp\left(-\frac{mu^2}{2k_B T}\right)\right] = 0 \tag{24}$$

Considering only the second bracket, i.e.,

$$\exp\left(-\frac{mu^2}{2k_b T}\right) u \left(2 - \frac{mu^2}{k_B T}\right) = 0. \tag{25}$$

and because the solution is $u \neq 0$, the only relevant term is the second one, i.e.,

$$2 - \frac{mu^2}{k_B T} = 0. \tag{26}$$

Therefore the most probable speed is

$$u_p = \sqrt{\frac{2k_B T}{m}} \tag{27}$$