Hydrodynamics and Magnetohydrodynamics: Solutions of the exercises in Lecture V

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Lecture V, Exercise 1.

ϵ, T, s, p, and ρ are the specific internal energy, the temperature, the specific entropy, the pressure and the rest-mass density respectively. $U = \epsilon m N$ is the internal energy and $S = smN$ is the entropy. The first law of thermodynamics is given by

$$\begin{aligned} dU = Nmd\epsilon &= TdS - pdV && (1) \\ &= TmNds - pdV && (2) \\ &= TmNds - pmNd\left(\frac{1}{\rho}\right), && (3) \end{aligned}$$

where we use $\rho = Nm/V,\ V = Nm/\rho \rightarrow dV = mNd(1/\rho)$. And the first law of thermodynamics is also written as

$$d\epsilon = Tds - p\left(\frac{1}{\rho}\right). \tag{4}$$

h is the specific enthalpy $h = (e+p)/\rho$, where e is energy density, $e = \rho + \rho\epsilon$. Taking derivative of enthalpy,

$$dh = d\left(\frac{e}{\rho}\right) + d\left(\frac{p}{\rho}\right). \tag{5}$$

From the definition of energy density,

$$e = \rho + \rho\epsilon \rightarrow \frac{e}{\rho} = 1 + \epsilon. \tag{6}$$

Taking derivative of eq (6)

$$d\left(\frac{e}{\rho}\right) = d\epsilon. \tag{7}$$

Using eq (7), the eq (5) can be written as

$$dh = d\epsilon + pd\left(\frac{1}{\rho}\right) + \frac{1}{\rho}dp. \tag{8}$$

Using eq (4) then we can obtain

$$\begin{aligned} dh &= Tds - pd\left(\frac{1}{\rho}\right) + pd\left(\frac{1}{\rho}\right) + \frac{1}{\rho}dp && (9) \\ &= Tds + \frac{1}{\rho}dp && (10) \end{aligned}$$

Therefore we can obtain the first law of thermodynamics with following from

$$dp = \rho dh - \rho T ds. \tag{11}$$

From the definition of energy density, we take a derivative and using eq (4) we obtain

$$\begin{aligned} de &= d\rho + d(\rho\epsilon) && (12)\\ &= d\rho + \rho d\epsilon + \epsilon d\rho && (13)\\ &= d\rho + \rho\left[T ds - p d\left(\frac{1}{\rho}\right)\right] + \epsilon d\rho. && (14) \end{aligned}$$

Where $pd(1/\rho) = -p/\rho^2 d\rho$,

$$\begin{aligned} de &= d\rho(1+\epsilon) + \rho T ds + \frac{p}{\rho} d\rho && (15)\\ &= \left(\frac{e}{\rho} + \frac{p}{\rho}\right) d\rho + \rho T ds && (16)\\ &= h d\rho + \rho T ds. && (17) \end{aligned}$$

We can obtain the first law of thermodynamics with following from

$$de = h d\rho + \rho T ds. \tag{18}$$

Lecture V, Exercise 2.

The entropy S and the enthalpy H have following relationship between the specific entropy and the specific enthalpy,

$$S = Nms, \quad H = Nmh. \tag{19}$$

Taking derivative in these quantities, we can obtain

$$dS = Nmds, \quad dH = Nmdh. \tag{20}$$

The density ρ can be given by

$$\rho = nm. \tag{21}$$

Using eq (20) and (21), eq (11) can be expressed as

$$\begin{aligned} dp &= \rho\left(\frac{1}{Nm} dH - T\frac{1}{Nm} dS\right) && (22)\\ &= \frac{1}{Nm} \cdot nm(dH - T dS) && (23)\\ &= \frac{n}{N}(dH - T dS). && (24) \end{aligned}$$

Therefore we can obtain the first law of thermodynamics with following from

$$dp = \frac{n}{N}(dH - TdS). \tag{25}$$

Taking derivative in eq (21) we can get

$$d\rho = mdn. \tag{26}$$

Using eq (20), (21) and (26), eq (18) can be written as

$$\begin{aligned} de &= \frac{H}{Nm} \cdot mdn + mnT \cdot \frac{1}{Nm}dS & (27) \\ &= \frac{1}{N}(Hdn + nTdS). & (28) \end{aligned}$$

Therefore we can obtain the first law of thermodynamics with following from

$$de = \frac{1}{N}(Hdn + nTdS). \tag{29}$$

Lecture V, Exercise 3.

The so-called Maxwell relations can be easy to obtain using the thermodynamic square (see Fig B.1 in textbook Relativistic Hydrodynamics).

The first law of thermodynamics can be expressed in four different forms, one for each side of square. For example, we consider the first law of thermodynamics written in the from eq (4), which corresponds left vertical side of the thermodynamic square. The differential $d\epsilon$ can be written in terms of differentials of $1/\rho$ and s which located the adjunct corners. To find the coefficients for these differentials follow diagonals to the opposite corner (use sense of the arrows to find the sign). Therefore from eq (4), when the density ρ is constant,

$$\left(\frac{\partial \epsilon}{\partial s}\right)_\rho = T. \tag{30}$$

And when the specific entropy s is constant,

$$\rho^2 \left(\frac{\partial \epsilon}{\partial \rho}\right)_s = p, \tag{31}$$

where we use $d(1/\rho) = -1/\rho^2 d\rho$.

Similarly, using the first law of thermodynamics written in the from eq (18), we change this form as

$$dh = \frac{1}{\rho}dp + Tds, \tag{32}$$

which corresponds in bottom side of the thermodynamic square. Using thermodynamic square, we can obtain following relations,

$$\left(\frac{\partial h}{\partial s}\right)_p = T, \quad \left(\frac{\partial h}{\partial p}\right)_s = \frac{1}{\rho}. \tag{33}$$

The same results can be calculated from eq. (32).

Next we consider right vertical side of the thermodynamic square. We follow the way to find the coefficients and can obtain

$$\left(\frac{\partial(\mu/m)}{\partial T}\right)_p = -s, \quad \left(\frac{\partial(\mu/m)}{\partial p}\right)_T = \frac{1}{\rho}. \tag{34}$$

The same results can be calculated from following form of the first law of thermodynamics

$$\frac{1}{\rho}dp = sdT + \frac{1}{m}d\mu. \tag{35}$$