Hydrodynamics and Magnetohydrodynamics: Solutions of the exercises in Lecture VI

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Lecture VI, Exercise 1.

The sound speed c_s is given by

$$c_s^2 = \left(\frac{\partial p}{\partial e}\right)_s. \tag{1}$$

We consider the first law of thermodynamics with following forms,

$$\begin{aligned} dp &= \rho dh - \rho T ds, & (2)\\ de &= h d\rho + \rho T ds & (3) \end{aligned}$$

Divide both equations and we get

$$\frac{dp}{de} = \frac{\rho}{h}\frac{dh}{d\rho}. \tag{4}$$

Therefore eq (1) can be written as

$$hc_s^2 = \rho\left(\frac{dh}{d\rho}\right) = \frac{dp}{d\rho}, \tag{5}$$

because $\rho dh = dp$ if $ds = 0$.

We consider the pressure p is a function of density and of the specific internal energy, $p = p(\rho, \epsilon)$. Taking derivative, we obtain

$$dp = \frac{\partial p}{\partial \rho} d\rho + \frac{\partial p}{\partial \epsilon} d\epsilon, \tag{6}$$

which can be divided by $d\rho$ to yield

$$\frac{dp}{d\rho} = \frac{\partial p}{\partial \rho} + \frac{\partial p}{\partial \epsilon}\frac{d\epsilon}{d\rho}. \tag{7}$$

From the first law of thermodynamics if $ds = 0$,

$$de = h d\rho. \tag{8}$$

Using following relations $e = \rho + \rho\epsilon$ and $h = (e + p)/\rho = 1 + \epsilon + p/\rho$, we obtain

$$d(\rho + \rho\epsilon) = \frac{e + p}{\rho} d\rho \tag{9}$$

$$d\rho + \rho d\epsilon + \epsilon d\rho = \frac{e + p}{\rho} d\rho \tag{10}$$

$$d\rho \left(1 + \epsilon - \frac{e + p}{\rho} \right) = -\rho d\epsilon \tag{11}$$

$$d\rho \left(\frac{\rho + \rho\epsilon - \rho - \rho\epsilon - p}{\rho} \right) = -\rho d\epsilon \tag{12}$$

$$\frac{d\epsilon}{d\rho} = \frac{p}{\rho^2}. \tag{13}$$

Adding Eqs. (6) and (13) to Eq. (5), the sound speed is written as

$$hc_s^2 = \frac{dp}{d\rho} = \left[\left(\frac{\partial p}{\partial \rho} \right)_s + \frac{p}{\rho^2} \left(\frac{\partial p}{\partial \epsilon} \right)_\rho \right]. \tag{14}$$

First we consider the ideal-fluid equation of state, $p = \rho\epsilon(\gamma - 1)$. We take a differential

$$dp = (\gamma - 1)(\rho d\epsilon + \epsilon d\rho), \tag{15}$$

and divide by $d\rho$,

$$\frac{dp}{d\rho} = (\gamma - 1) \left[\rho \frac{d\epsilon}{d\rho} + \epsilon \right]. \tag{16}$$

From the definition of energy density $e = \rho + \rho\epsilon$, we take a derivative and using the first law of thermodynamics,

$$\begin{aligned} de &= d\rho + \rho d\epsilon + \epsilon d\rho && (17) \\ &= (1 + \epsilon) d\rho + \rho d\epsilon = h d\rho, && (18) \end{aligned}$$

we rewrite it as

$$1 + \epsilon + \rho \frac{d\epsilon}{d\rho} = h = 1 + \gamma\epsilon \tag{19}$$

$$\epsilon + \rho \frac{d\epsilon}{d\rho} = \gamma\epsilon. \tag{20}$$

Adding Eq. (20) to Eq. (16) we obtain

$$\frac{dp}{d\rho} = (\gamma - 1)[\epsilon(\gamma - 1) + \epsilon] = (\gamma - 1)\gamma\epsilon = \frac{\gamma p}{\rho}. \tag{21}$$

Therefore the square of the sound speed using ideal-fluid equation of state is written as

$$c_s^2 = \frac{1}{h} \frac{dp}{d\rho} = \frac{\gamma p}{\rho h} = \frac{(\gamma - 1)\gamma\epsilon}{1 + \gamma\epsilon} = (h - 1)(\gamma - 1) \tag{22}$$

Second we consider the polytropic equation of state, $p = K\rho^\Gamma$. Taking a differential we obtain

$$dp = \left(\frac{\Gamma p}{\rho}\right) d\rho \tag{23}$$

The energy density for polytropic equation of state is written as

$$e = \rho + \frac{1}{\Gamma - 1} p = \rho + \rho\epsilon. \tag{24}$$

Using Eqs. (23) and (24), the square of sound speed using the polytropic equation of state is obtained as

$$\begin{aligned} c_s^2 &= \frac{1}{h}\frac{dp}{d\rho} = \frac{\Gamma p}{\rho h} = \frac{\Gamma p}{\rho + \rho\epsilon + p} && (25) \\ &= \frac{\Gamma p}{\rho + \frac{p}{\Gamma - 1} + p} = \frac{\Gamma(\Gamma - 1)p}{\rho(\Gamma - 1) + p\Gamma}. && (26) \end{aligned}$$

Lecture VI, Exercise 2.

The pressure has the following relation,

$$p = \rho\epsilon(\gamma - 1) = nm\epsilon(\gamma - 1) = nk_B T \tag{27}$$

Therefore the temperature is given by

$$T = \frac{m}{k_B}(\gamma - 1)\epsilon. \tag{28}$$

From the first law of thermodynamics,

$$d\epsilon = T ds + \frac{p}{\rho^2} d\rho. \tag{29}$$

Using Eq. (28), it can be rewritten as

$$\begin{aligned} ds &= \frac{1}{T} d\epsilon - \frac{p}{\rho^2 T} d\rho && (30) \\ &= \frac{k_B}{m(\gamma - 1)\epsilon} d\epsilon - \frac{p k_B}{\rho^2 m(\gamma - 1)\epsilon} d\rho. && (31) \end{aligned}$$

Using Eq. (27), Eq (31) is also written as

$$\begin{aligned} \frac{m}{k_B} ds &= \frac{d\epsilon}{\epsilon(\gamma - 1)} - \frac{d\rho}{\rho} && (32) \\ &= \frac{d\ln\epsilon}{\gamma - 1} - d\ln\rho && (33) \\ &= d\ln\epsilon^{1/\gamma - 1} - d\ln\rho && (34) \\ &= d\left[\ln\left(\frac{\epsilon^{1/\gamma - 1}}{\rho}\right)\right]. && (35) \end{aligned}$$

We can now integrate Eq. (35) to obtain

$$s = \frac{k_B}{m}\left[\ln\left(\frac{\epsilon^{1/\gamma-1}}{\rho}\right)+\tilde{K}\right]. \tag{36}$$

Here we consider the polytropic equation of state ($p = K\rho^{\Gamma}$). The specific internal energy is given by

$$\epsilon = \frac{K\rho^{\Gamma-1}}{\Gamma-1}. \tag{37}$$

Thus

$$\frac{\epsilon^{1/\Gamma-1}}{\rho} = \left(\frac{K}{\Gamma-1}\right)^{1/\Gamma-1}\frac{\rho}{\rho} = \left(\frac{K}{\Gamma-1}\right)^{1/\Gamma-1}. \tag{38}$$

As a result, Eq. (36) can be written as

$$s = \frac{k_B}{m}\left[\ln\left(\frac{K}{\Gamma-1}\right)^{1/\Gamma-1}+\tilde{K}\right]. \tag{39}$$

Lecture VI, Exercise 3.

Let's start from the first law of thermodynamics

$$dp = \rho dh - \rho T ds. \tag{40}$$

Here we consider polytropic equation of state which pressure is a function of density only ($p = p(\rho)$). Therefore

$$dp = \frac{\partial p}{\partial \rho}d\rho = P'_{\rho}d\rho. \tag{41}$$

The specific enthalpy $h = e + p/\rho = 1 + \epsilon + p/\rho$. Taking the differential we obtain

$$dh = d\epsilon + d\left(\frac{p}{\rho}\right). \tag{42}$$

From the polytropic equation of state, we know that the pressure is a function of density only, so that the internal energy is a function of density only ($\epsilon = \epsilon(\rho)$). Thus,

$$d\epsilon = \frac{\partial \epsilon}{\partial \rho}d\rho = \epsilon'_{\rho}d\rho. \tag{43}$$

Using Eq (41), the second term of RHS in Eq (42) can be expressed as

$$d\left(\frac{p}{\rho}\right) = \frac{1}{\rho}dp - \frac{p}{\rho^2}d\rho = \frac{p'_{\rho}}{\rho}d\rho - \frac{p}{\rho^2}d\rho = \frac{d\rho}{\rho}\left(p'_{\rho} - \frac{p}{\rho}\right). \tag{44}$$

Therefore Eq. (40) is given by

$$
\begin{aligned}
& p'_\rho d\rho = \rho \left[\epsilon'_\rho d\rho + \frac{d\rho}{\rho} \left(p'_\rho - \frac{p}{\rho} \right) \right] - \rho T ds && (45) \\
\rightarrow \quad & 0 = \rho \epsilon'_\rho d\rho - \frac{p}{\rho} d\rho - \rho T ds && (46) \\
\rightarrow \quad & d\rho \left(\epsilon'_\rho - \frac{p}{\rho^2} \right) = T ds && (47) \\
\rightarrow \quad & d\rho \left(\frac{\partial \epsilon}{\partial \rho} - \frac{p}{\rho^2} \right) = T ds. && (48)
\end{aligned}
$$

This equation shows that if $\partial\epsilon/\partial\rho = p/\rho^2$, the polytropic equation of state is isentropic $(ds = 0)$.