Hydrodynamics and Magnetohydrodynamics: Solutions of the exercises in Lecture VII

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Lecture VII, Exercise 1.

u^μ is 4-velocity and a^μ is 4-acceleration, whose contravariant components are defined as

$$a^\mu = u^\nu \nabla_\nu u^\mu. \tag{1}$$

There is a normalization condition

$$u^\mu u_\mu = -1 \tag{2}$$

and the orthogonality condition

$$a^\mu u_\mu = 0. \tag{3}$$

After taking a contravariant derivative of Eq (2), we can obtain the identity

$$u^\mu \nabla_\nu u_\mu = 0. \tag{4}$$

First we consider the expansion scalar Θ. Starting from its definition, the expansion scalar can be rewritten as

$$
\begin{aligned}
\Theta &= h^{\mu\nu} \nabla_\nu u_\mu & (5) \\
&= (g^{\mu\nu} + u^\mu u^\nu) \nabla_\nu u_\mu & (6) \\
&= g^{\mu\nu} \nabla_\nu u_\mu + u^\mu u^\nu \nabla_\nu u_\mu & (7) \\
&= \nabla_\nu g^{\mu\nu} u_\mu \quad \text{(here using Eq (4))} & (8) \\
&= \nabla_\nu u^\nu. & (9)
\end{aligned}
$$

Next we consider the kinematic vorticity tensor $\omega_{\mu\nu}$. Starting from its definition,

the kinematic vorticity tensor can be rewritten as

$$
\begin{aligned}
\omega_{\mu\nu} &= h_\mu^\alpha h_\nu^\beta \nabla_{[\beta, u_\alpha]} \tag{10}\\
&= \frac{1}{2}(g_\mu^\alpha + u^\alpha u_\mu)(g_\nu^\beta + u^\beta u_\nu)(\nabla_\beta u_\alpha - \nabla_\alpha u_\beta) \tag{11}\\
&= \frac{1}{2}(g_\mu^\alpha + u^\alpha u_\mu)[g_\nu^\beta(\nabla_\beta u_\alpha - \nabla_\alpha u_\beta) + u^\beta u_\nu \nabla_\beta u_\alpha - u^\beta u_\nu \nabla_\alpha u_\beta] \tag{12}\\
&= \frac{1}{2}(g_\mu^\alpha + u^\alpha u_\mu)[(\nabla_\nu u_\alpha - \nabla_\alpha u_\nu) + u^\beta u_\nu \nabla_\beta u_\alpha] \quad \text{(using Eq (4))} \tag{13}\\
&= \frac{1}{2}g_\mu^\alpha(\nabla_\nu u_\alpha - \nabla_\alpha u_\nu) + \frac{1}{2}[u^\alpha u_\mu(\nabla_\nu u_\alpha - \nabla_\alpha u_\nu)]\\
&\quad + \frac{1}{2}g_\mu^\alpha u_\nu u^\beta \nabla_\beta u_\alpha + \frac{1}{2}u^\alpha u_\mu u^\beta u_\nu \nabla_\beta u_\alpha \tag{14}\\
&= \frac{1}{2}(\nabla_\nu u_\mu - \nabla_\mu u_\nu) + \frac{1}{2}(u^\beta \nabla_\beta u_\mu u_\nu - u^\alpha \nabla_\alpha u_\nu u_\mu) \quad \text{(using Eq (4))} \tag{15}\\
&= \frac{1}{2}(\nabla_\nu u_\mu - \nabla_\mu u_\nu) + \frac{1}{2}(a_\mu u_\nu - a_\nu u_\mu) \quad \text{(using Eq (1))} \tag{16}\\
&= \nabla_{[\nu u_\mu]} + a_{\mu} u_\nu. \tag{17}
\end{aligned}
$$

Next we consider the shear tensor $\sigma_{\mu\nu}$. Starting from its definition, the shear tensor can be rewritten as

$$
\begin{aligned}
\sigma_{\mu\nu} &= \nabla_{\langle \mu u_\nu \rangle} \tag{18}\\
&= h_\mu^\alpha h_\nu^\beta \nabla_{(\beta u_\alpha)} - \frac{1}{3}\nabla_\beta u_\alpha h^{\alpha\beta} h_{\mu\nu} \tag{19}\\
&= \frac{1}{2}(g_\mu^\alpha + u^\alpha u_\mu)(g_\nu^\beta + u^\beta u_\nu)(\nabla_\beta u_\alpha + \nabla_\alpha u_\beta) - \frac{1}{3}\Theta h_{\mu\nu} \tag{20}\\
&\quad \text{(using Eq (5))}\\
&= \frac{1}{2}(g_\mu^\alpha + u^\alpha u_\mu)[g_\nu^\beta(\nabla_\beta u_\alpha + \nabla_\alpha u_\beta) + u^\beta u_\nu(\nabla_\beta u_\alpha + \nabla_\alpha u_\beta)]\\
&\quad - \frac{1}{3}\Theta h_{\mu\nu} \tag{21}\\
&= \frac{1}{2}(g_\mu^\alpha + u^\alpha u_\mu)[(\nabla_\nu u_\alpha + \nabla_\alpha u_\nu) + u^\beta u_\nu \nabla_\beta u_\alpha] - \frac{1}{3}\Theta h_{\mu\nu} \tag{22}\\
&\quad \text{(using Eq (4))}\\
&= \frac{1}{2}g_\mu^\alpha(\nabla_\nu u_\alpha + \nabla_\alpha u_\nu) + \frac{1}{2}g_\mu^\alpha u^\beta u_\nu \nabla_\beta u_\alpha\\
&\quad + \frac{1}{2}u^\alpha u_\mu(\nabla_\nu u_\alpha + \nabla_\alpha u_\nu) + \frac{1}{2}u^\alpha u_\mu u^\beta u_\nu \nabla_\beta u_\alpha + \frac{1}{3}\Theta h_{\mu\nu} \tag{23}\\
&= \frac{1}{2}(\nabla_\nu u_\mu + \nabla_\mu u_\nu) + \frac{1}{2}(a_\mu u_\nu + a_\nu u_\mu) - \frac{1}{3}\Theta h_{\mu\nu} \tag{24}\\
&\quad \text{(using Eq (4) \& (5))}\\
&= \nabla_{(\mu u_\nu)} + a_{(\mu u_\nu)} - \frac{1}{3}\Theta h_{\mu\nu}. \tag{25}
\end{aligned}
$$

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Lecture VII, Exercise 2.

The energy-momentum tensor of a perfect fluid can be written as

$$T^{\mu\nu} = (e+p)u^\mu u^\nu + pg^{\mu\nu}. \tag{26}$$

Now we consider the following projection

$$
\begin{aligned}
L_\mu &= -h_\mu^\alpha u^\beta T_{\alpha\beta} & (27) \\
&= -(g_\mu^\alpha + u^\alpha u_\mu)[(e+p)u^\beta u_\alpha u_\beta + pg_{\alpha\beta}u^\beta] & (28) \\
&= -(e+p)g_\mu^\alpha u^\beta u_\alpha u_\beta - pg_{\alpha\beta}g_\mu^\alpha u^\beta \\
&\quad -(e+p)u^\alpha u_\mu u^\beta u_\alpha u_\beta - pg_{\alpha\beta}u^\beta u^\alpha u_\mu & (29) \\
&= (e+p)g_\mu^\alpha u_\alpha - (e+p)u_\alpha u_\alpha u^\beta u_\beta u_\mu \\
&\quad -pg_\mu^\alpha g_{\alpha\beta}u^\beta - u^\alpha u_\mu p u_\alpha & (30) \\
&= (e+p)u_\mu - (e+p)u_\mu - pu_\mu + pu_\mu & (31) \\
&= 0. & (32)
\end{aligned}
$$

This projection is identically zero for the case of a perfect fluid. Because in the case of a perfect fluid, the energy-momentum tensor in the local rest frame of a comoving observer is symmetric. Since there are no off-diagonal components, the spatial momentum density is identically zero.

Lecture VII, Exercise 3.

The spatial stress tensor $L_{\mu\nu}$, the spatial momentum density L_μ, and the energy density e can be defined using the energy-momentum tensor $T_{\alpha\beta}$ and the projection

$$
\begin{aligned}
L_{\mu\nu} &= h_\mu^\alpha h_\nu^\beta T_{\alpha\beta}, & (33) \\
L_\mu &= -h_\mu^\alpha u^\beta T_{\alpha\beta}, & (34) \\
e &= u^\alpha u^\beta T_{\alpha\beta}. & (35)
\end{aligned}
$$

We try to show the following identity is true

$$T_{\mu\nu} = eu_\mu u_\nu + 2u_{(\mu}L_{\nu)} + L_{\mu\nu}. \tag{36}$$

In Eq (36), the each terms are given by

$$
\begin{aligned}
eu_\mu u_\nu &= u^\alpha u^\beta T_{\alpha\beta} u_\mu u_\nu, & (37) \\
2u_{(\mu}L_{\mu)} &= 2 \times \frac{1}{2}(u_\mu L_\nu + u_\nu L_\mu) = -h_\nu^\alpha u^\beta T_{\alpha\beta} u_\mu - h_\mu^\alpha u^\beta T_{\alpha\beta} u_\nu, & (38) \\
L_{\mu\nu} &= h_\mu^\alpha h_\nu^\beta T_{\alpha\beta}. & (39)
\end{aligned}
$$

Therefore

$$
\begin{array}{rll}
\text{Eq (36)} & = & T_{\alpha\beta}(u^\alpha u^\beta u_\mu u_\nu - h^\alpha_\nu u^\beta u_\mu - h^\alpha_\mu u^\beta u_\mu + h^\alpha_\mu h^\beta_\nu) \qquad (40) \\[4pt]
& = & T_{\alpha\beta}[u^\alpha u^\beta u_\mu u_\nu - (g^\alpha_\nu + u^\alpha u_\nu)u^\beta u_\mu - (g^\alpha_\mu + u^\alpha u_\mu)u^\beta u_\nu \\[4pt]
& & +(g^\alpha_\mu + u^\alpha u_\mu)(g^\beta_\nu + u^\beta u_\nu)] \qquad (41) \\[4pt]
& = & T_{\alpha\beta}(u^\alpha u^\beta u_\mu u_\nu - g^\alpha_\nu u^\beta u_\mu - u^\alpha u_\nu u^\beta u_\mu - g^\alpha_\mu u^\beta u_\nu - u^\alpha u_\mu u^\beta u_\nu \\[4pt]
& & +g^\alpha_\mu g^\beta_\nu + g^\alpha_\mu u^\beta u_\nu + g^\beta_\nu u^\alpha u_\mu + u^\alpha u_\mu u^\beta u_\nu) \qquad (42) \\[4pt]
& = & T_{\alpha\beta}(-g^\alpha_\nu u^\beta u_\mu - g^\alpha_\mu u^\beta u_\nu + g^\alpha_\mu g^\beta_\nu + g^\alpha_\mu u^\beta u_\nu + g^\beta_\nu u^\alpha u_\mu) \qquad (43) \\[4pt]
& = & T_{\alpha\beta} g^\alpha_\mu g^\beta_\nu \qquad (44) \\[4pt]
& = & T_{\mu\nu}. \qquad (45)
\end{array}
$$

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