Hydrodynamics and Magnetohydrodynamics: Solutions of the exercises in Lecture VIII

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Lecture VIII, Exercise 1.

We consider the Newtonian limit of the hydrodynamic equations. For this purpose we assume that the gravitational field with potential ϕ which is solution of the Poisson equation $\vec{\nabla}^2\phi = 4\pi G\rho$ and that is weak and static, so that it is possible to find a coordinate system $x^\alpha = (t, x^i)$ such that the metric components take the form (note that we will retain explicitly the speed of light, c, as this will help us keep track of the order of different terms in an expansion in terms of the normalized fluid velocity v/c)

$$ds^2 = -\left(1 + 2\frac{\phi}{c^2}\right)c^2dt^2 + \left(1 - 2\frac{\phi}{c^2}\right)\eta_{ij}dx^idx^j, \tag{1}$$

where η_{ij} is the flat three-metric. If we define as $v^i = dx^i/dt$ the components of the fluid velocity $\vec{v}$ with respect to the inertial frame defined by the coordinate x^α in Eq. (1) (these coordinates become inertial in the limit $\phi/c^2 \to 0$), then the Newtonian limit is obtained by additionally requiring that $|\vec{v}|/c \ll 1$. Using the normalization condition for the four-velocity and the expression for the four-velocity components, we can express u^0 in terms of ϕ and $\vec{v}$. To first order in ϕ and v_jv^j, we can write

$$u^0 \simeq 1 - \frac{\phi}{c^2} + \frac{1}{2}\frac{v_jv^j}{c^2}, \quad u^0\frac{v^i}{c} \simeq \frac{v^i}{c}\mathcal{O}\left(\frac{v^iv_jv^j}{c^3}\right). \tag{2}$$

As a result, the contravariant components of the four-velocity vector in the Newtonian limit are given by

$$u^\alpha \simeq \left(u^0, \frac{v^i}{c}\right) = \left(1 - \frac{\phi}{c^2} + \frac{1}{2}\frac{v_jv^j}{c^2}, \frac{v^i}{c}\right), \tag{3}$$

while the corresponding covariant components are given by

$$u_\alpha \simeq \left(u_0, \frac{v_i}{c}\right) = \left(-1 - \frac{\phi}{c^2} - \frac{1}{2}\frac{v_jv^j}{c^2}, \frac{v_i}{c}\right). \tag{4}$$

Next we can consider the Newtonian limit of the relativistic continuity equation

$$u^\mu\nabla_\mu\rho + \rho\nabla_\mu u^\mu = 0. \tag{5}$$

The operator $u^\mu\nabla_\mu$ reduces to the convective derivative in the Newtonian framework, i.e.,

$$u^\mu\nabla_\mu \to \frac{D}{Dt} := \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}, \tag{6}$$

since

$$u^\mu \nabla_\mu = \frac{u^0}{c}\frac{\partial}{\partial t} + u^i \frac{\partial}{\partial x^i}, \tag{7}$$

and we can use expressions (3) and (4) to rewrite the differential operators at $\mathcal{O}(v^2/c^2)$ as

$$\frac{u^o}{c}\frac{\partial}{\partial t} \simeq \frac{1}{c}\frac{\partial}{\partial t}, \tag{8}$$

$$u^i \frac{\partial}{\partial x^i} \simeq \frac{v^i}{c}\frac{\partial}{\partial x^i}. \tag{9}$$

Similarly, the term $\nabla_\mu u^\mu$ can be expressed by

$$\nabla_\mu u^\mu = \frac{1}{c}\frac{\partial u^0}{\partial t} + \frac{\partial u^i}{\partial x^i} \simeq \frac{1}{c}\frac{\partial v^i}{\partial x^i}, \tag{10}$$

Collecting all these results, the Newtonian limit of Eq. (5) is given by

$$\frac{D}{Dt}\rho + \rho(\vec{\nabla}\cdot\boldsymbol{v}) \tag{11}$$

$$= \partial_t \rho + v^i \partial_i \rho + \rho \partial_i v^i = 0. \tag{12}$$

Lecture VIII, Exercise 2.

Here we can consider the Newtonian limit of the relativistic equation of momentum conservation

$$u^\mu \nabla_\mu u_\nu + \frac{1}{\rho h} h^\mu_\nu \nabla_\mu p = 0. \tag{13}$$

Here we need to introduce the concept of external forces. The typical example of an external force is represented by the gravitational force, the electromagnetic forces (which act when the fluid has the net electromagnetic charge), and the fictitious forces (such as the centrifugal or Coriolis force), which appear when the fluid motion is described in a non-inertial (e.g., rotating) reference frame. The important difference between Newtonian description and relativistic ones is that, if present, the gravitational forces (and the possible fictitious forces) are no longer external forces but are accounted for by the curvature of spacetime.

In the Newtonian description, we consider $\epsilon \ll c^2$ (i.e., the energy density of the fluid is essentially given by the rest-mass energy) and $\rho/p \ll c^2$ (i.e., the pressure contribution to the energy density is negligible). Hence the Newtonian limit of the specific enthalpy and of the second term in Eq. (13) are given respectively by

$$h = c^2 + \epsilon + \frac{p}{\rho} \quad \to \quad c^2, \tag{14}$$

$$\frac{1}{\rho h} h^\mu_\nu \nabla_\mu p \quad \to \quad \frac{1}{\rho c^2}\left(\partial_j p + \frac{v_j}{c^2}\partial_t p + \frac{v_j v^j}{c^2}\partial_j p\right) \simeq \frac{1}{\rho c^2}\partial_j p. \tag{15}$$

Collecting these results and considering the presence of an external force of gravitational origin, the Newtonian limit of Eq. (13) is given by

$$\partial_t v^i + v^j \partial_j v^i + \frac{1}{\rho}\partial_i p + \partial_i \phi = 0 \tag{16}$$

Lecture VIII, Exercise 3.

Here we can consider the Newtonian limit of the relativistic equation of energy conservation

$$u^\mu \nabla_\mu e + \rho h \nabla_\mu u^\mu = 0. \tag{17}$$

Similarly we consider following change of each term in the Newtonian limit

$$u^\mu \nabla_\mu \to D_t, \quad \nabla_\mu u^\mu \to \vec{\nabla} \cdot \vec{\boldsymbol{v}}, \tag{18}$$

$$e \to \rho + \rho\epsilon, \quad \rho h \to \rho + \rho\epsilon + p. \tag{19}$$

Thus, Eq. (17) can be written as

$$\begin{aligned} & D_t(\rho + \rho\epsilon) + (\rho + \rho\epsilon + p)\vec{\nabla} \cdot \vec{\boldsymbol{v}} = 0 \\ \Rightarrow \quad & \partial_t \rho + \epsilon \partial_t \rho + p \partial_t \epsilon + \vec{\boldsymbol{v}} \cdot \vec{\nabla}\rho + \vec{\boldsymbol{v}} \cdot \vec{\nabla}(\rho\epsilon) + (\rho + \rho\epsilon)\vec{\nabla} \cdot \vec{\boldsymbol{v}} = 0. \end{aligned} \tag{20}$$

From the continuity equation

$$\begin{aligned} & \partial_t \rho + \vec{\boldsymbol{v}} \cdot \vec{\nabla}\rho = -\rho \vec{\nabla} \cdot \vec{\boldsymbol{v}} \\ \Rightarrow \quad & \epsilon(\partial_t \rho + \vec{\boldsymbol{v}} \cdot \vec{\nabla}\rho) = -\rho\epsilon \vec{\nabla} \cdot \vec{\boldsymbol{v}}. \end{aligned} \tag{21}$$

Using Eq. (20), Eq. (21) can be written as

$$\begin{aligned} & -\rho \vec{\nabla} \cdot \vec{\boldsymbol{v}} - \rho\epsilon \vec{\nabla} \cdot \vec{\boldsymbol{v}} + \rho \partial_t \epsilon + \rho \vec{\boldsymbol{v}} \cdot \vec{\nabla}\epsilon + \rho \vec{\vec{\nabla}} \vec{\boldsymbol{v}} \\ & + \rho\epsilon \vec{\nabla} \cdot \vec{\boldsymbol{v}} + p \vec{\nabla} \cdot \vec{\boldsymbol{v}} = 0 \\ \Rightarrow \quad & \rho(\partial_t \epsilon + \vec{\boldsymbol{v}} \cdot \vec{\nabla}\epsilon) + p \vec{\nabla} \cdot \vec{\boldsymbol{v}} = 0 \\ \Rightarrow \quad & \rho D_t \epsilon + p \vec{\nabla} \cdot \vec{\boldsymbol{v}} = 0. \end{aligned} \tag{22}$$

Let's remain within a Newtonian framework and let's rewrite Eq. (22) as

$$\begin{aligned} & \rho D_t \epsilon + p \vec{\nabla} \cdot \vec{\boldsymbol{v}} + \epsilon(D_t \rho + \rho \vec{\nabla} \cdot \vec{\boldsymbol{v}}) = 0 \\ \Rightarrow \quad & D_t(\rho\epsilon) + \rho\epsilon \vec{\nabla} \cdot \vec{\boldsymbol{v}} + p \vec{\nabla} \cdot \vec{\boldsymbol{v}} = 0 \quad \text{(using the continuity equation)} \\ \Rightarrow \quad & D_t(\rho\epsilon) + (\rho\epsilon + p)\vec{\nabla} \cdot \vec{\boldsymbol{v}} = 0 \\ \Rightarrow \quad & \partial_t(\rho\epsilon) + \vec{\nabla} \cdot (\rho\epsilon \vec{\boldsymbol{v}}) + p \vec{\nabla} \cdot \vec{\boldsymbol{v}} = 0 \\ \Rightarrow \quad & (\text{add } \pm \vec{\boldsymbol{v}} \cdot \vec{\nabla} p)) \\ \Rightarrow \quad & \partial_t(\rho\epsilon) + \vec{\nabla} \cdot [(\rho\epsilon + p)\vec{\boldsymbol{v}}] - \vec{\boldsymbol{v}} \cdot \vec{\nabla} p = 0. \end{aligned} \tag{23}$$

Here we consider following equation

$$
\begin{aligned}
& \partial_t(\rho v^2) + \vec{\nabla}\cdot(\rho v^2 \vec{\boldsymbol{v}}) \\
= \; & \rho \partial_t v^2 + v^2 \partial_t \rho + v^2 \vec{\nabla}(\rho \vec{\boldsymbol{v}}) + \rho \vec{\boldsymbol{v}} \cdot \vec{\nabla} v^2 \\
= \; & \rho(\partial_t v^2 + \vec{\boldsymbol{v}} \cdot \vec{\nabla} v^2) \qquad (24)
\end{aligned}
$$

Then using the following relation

$$
\begin{aligned}
\partial_t v^2 &= 2\vec{\boldsymbol{v}} \cdot \partial_t \vec{\boldsymbol{v}}, \qquad (25) \\
\vec{\boldsymbol{v}} \cdot \vec{\nabla} v^2 &= 2\vec{\boldsymbol{v}} \cdot (\vec{\boldsymbol{v}} \cdot \vec{\nabla} \vec{\boldsymbol{v}}), \qquad (26)
\end{aligned}
$$

Eq. (24) can be expressed as

$$
\begin{aligned}
= \; & 2\rho \vec{\boldsymbol{v}}(\partial_t \vec{\boldsymbol{v}} + \vec{\boldsymbol{v}} \cdot \vec{\nabla} \vec{\boldsymbol{v}}) \\
= \; & 2\rho \vec{\boldsymbol{v}} \cdot D_t \vec{\boldsymbol{v}} = 2\rho \vec{\boldsymbol{v}} \left(-\frac{1}{\rho} \vec{\nabla} p - \vec{\nabla} \phi \right) \\
= \; & -2\vec{\boldsymbol{v}} \vec{\nabla} p - 2\rho \vec{\boldsymbol{v}} \cdot \vec{\nabla} \phi, \qquad (27)
\end{aligned}
$$

where we have used the equation of momentum conservation. As a result we obtain

$$
-\vec{\boldsymbol{v}} \cdot \vec{\nabla} p = \partial_t \left(\frac{1}{2} \rho v^2 \right) + \vec{\nabla} \cdot \left(\frac{\rho v^2}{2} \vec{\boldsymbol{v}} \right) + \rho \vec{\boldsymbol{v}} \cdot \vec{\nabla} \phi. \qquad (28)
$$

Using this relation in Eq. (23), we finally obtain the Newtonian limit of the energy conservation equation

$$
\partial_t \left(\frac{1}{2} \rho v^2 + \rho \epsilon \right) + \vec{\nabla} \cdot \left[\left(\frac{1}{2} \rho v^2 + \rho \epsilon + p \right) \vec{\boldsymbol{v}} \right] = -\rho \vec{\boldsymbol{v}} \cdot \vec{\nabla} \phi. \qquad (29)
$$

Lecture VIII, Exercise 4.

The vorticity tensor is defined as

$$
\begin{aligned}
\Omega_{\mu\nu} &= 2\nabla_{[\mu}\omega_{\nu]} \\
&= \nabla_\nu(h u_\mu) - \nabla_\mu(h u_\nu) \\
&= h\nabla_\nu u_\mu + u_\mu \nabla_\nu h - h\nabla_\mu u_\nu - u_\nu \nabla_\mu h \\
&= h(\nabla_\nu u_\mu - \nabla_\mu u_\nu) + u_\mu \nabla_\nu h - u_\nu \nabla_\mu h. \qquad (30)
\end{aligned}
$$

The kinematic vorticity tensor is defined as

$$
\begin{aligned}
\omega_{\mu\nu} &= h^\alpha_\mu h^\beta_\nu \nabla_{[\beta} u_{\alpha]} \\
&= \nabla_{[\mu} u_{\nu]} + a_{[\mu} u_{\nu]} \\
&= \frac{1}{2}(\nabla_\nu u_\mu - \nabla_\mu u_\nu) + a_{[\mu} u_{\nu]}. \qquad (31)
\end{aligned}
$$

Thus,

$$\nabla_\nu u_\mu - \nabla_\mu u_\nu = 2(\omega_{\mu\nu} - a_{[\mu} u_{\nu]}). \tag{32}$$

Substituting Eq (34) into Eq (32) we obtain

$$\begin{aligned}
\Omega_{\mu\nu} &= 2h(\omega_{\mu\nu} - a_{[\mu} u_{\nu]}) + u_\mu \nabla_\nu h - u_\nu \nabla_\mu h \\
&= 2h \left[\omega_{\mu\nu} - a_{[\mu} u_{\nu]} + \frac{1}{2} \left(u_\mu \frac{1}{h} \nabla_\nu h - u_\nu \frac{1}{h} \nabla_\mu h \right) \right] \\
&= 2h[\omega_{\mu\nu} - a_{[\mu} u_{\nu]} + u_{[\mu} \nabla_{\nu]} \ln h].
\end{aligned} \tag{33}$$

From the equation above it is clear that only for a test fluid (i.e., $e = 0 = p$ and $h = 1$) in geodetic motion (i.e., $a_\mu = 0$) two tensors are directly proportional, $\Omega_{\mu\nu} = 2\omega_{\mu\nu}$.